Statistics for Astronomers: Lecture 2, 2020.09.23

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BY NC SA

Review

Probability: Classical vs. Frequentist vs. Bayesian.
Kolmogorov Axioms.
Conditional and marginal probability.
Independence and exclusivity.
Law of Total Probability from union of pairwise disjoint, collectively exhaustive sets.
Bayes' Theorem.

Random variables and probability distributions

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Convention:
X: random variable, x: value assigned to random variable.
"Probability that X has value x": $P(X = x)$, $P_X(x)$ (x discrete) or $p_X(x)$ (x continuous).
Specifying $P_X(x)$ or $p_X(x)$ for all $x \in S$, the state space results in a probability distribution.
Discrete: mass function (PMF). Continuous: density function (PDF).
Note: I won't abbreviate "probability distribution function", so that "PDF" is unambiguous.

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$$P_{x_3}(x_3) = \sum_{x_2 \in S} \sum_{x_1 \in S} P(X_3 = x_3 | X_2 = x_2, X_1 = x_1)\ P(X_2 = x_2 | X_1 = x_1)\ P(X_1 = x_1)$$

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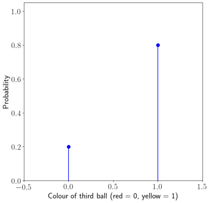

Code for plot available ▶ here.

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Sum of numbers displayed on two dice after one throw (discrete distribution):

State space $= \{2, 3, ..., 11, 12\}$

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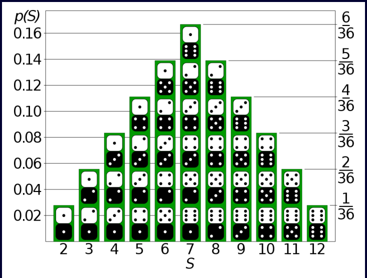

(Tim Stellmach/Public Domain)

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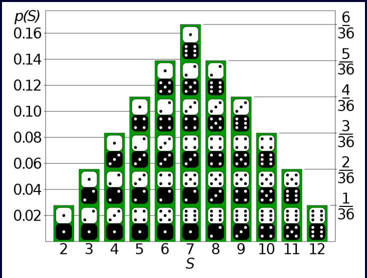

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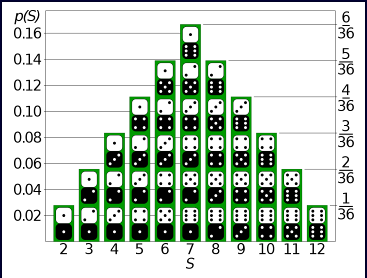

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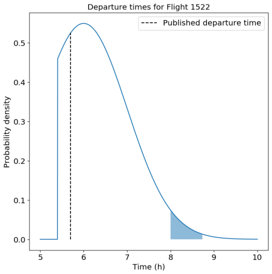
Actual departure time of Flight 1522 (continuous distribution):

State space $= (5.4, \infty)$

Code for plot available here.



Shaded region $= P(8{:}00\ \mathrm{AM} < T < 8{:}44\ \mathrm{AM})$.

$P(X = 8{:}15\ \mathrm{AM})$ is meaningless, zero.

$$P(8{:}00\ \mathrm{AM} < T < 8{:}44\ \mathrm{AM}) = \int_{8:00\ \mathrm{AM}}^{8:44\ \mathrm{AM}} p_X(x)\, dx \text{ is finite.}$$

Populations and samples

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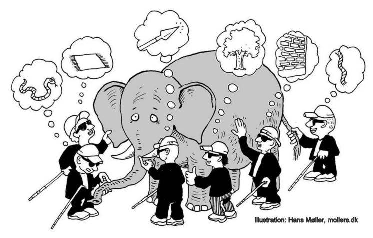

"More data is required."

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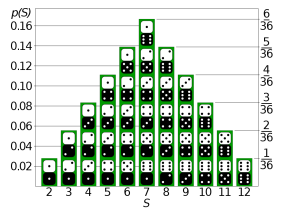

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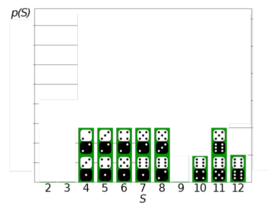
Population: $\mu = 7.00$, $Var(X) = 5.83$.



Sample of outcomes obtained from rolling two dice 14 times.
$\bar{x} = 7.43$, sample variance (discussed later) $= 6.67$

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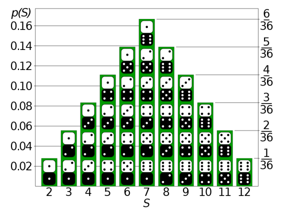

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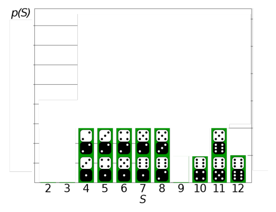

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Sample distribution seems uniform, results in a larger sample mean/variance than the true (population) values.

Cumulative distribution function (CDF)

Definition (Cumulative distribution function)

A function $F_X(x)$ of a random variable X such that $F_X(x)$ is the probability that $X \leq x$.

For a discrete PMF:

$$F_X(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i).$$

For a continuous PDF:

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From this definition, the probability of the variable ranging between two values a and b is

$P(a < X \leq b) = F_X(x = b) - F_X(x = a)$. For a PDF, this is also equal to $\int_{t=a}^{t=b} p_X(t)\ dt$.

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The CDF is a monotonically increasing function.

For a discrete random variable, it is constant in between values.

For the continuous case, the PDF is the derivative of the CDF w.r.t. x.

Cumulative distribution function (contd.)

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Definition (Independent and identically distributed variables)

Two random variables X and Y are said to be iid if and only if they are mutually independent and drawn from the same distribution:

$$F_{X,Y}(x, y) = F_X(x) \times F_Y(y)$$
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Heteroskedastic: originally iid observations + measurement errors that aren't identical.

Typical case in astronomy.

Expectation value

Definition (Expectation value)

The expectation value $E[g(X)]$ of a function $g(X)$ of a random variable X, is the weighted average of $g(X)$, with the weights being the associated probabilities:

Discrete: $E[g(X)] = \sum_{i=1}^{N} g(x_i)\ P(X = x_i)$.

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Linearity: $E[\alpha g(X) + \beta h(X)] = \alpha E[g(X)] + \beta E[h(X)]$. ($\sum$ and $\int$ are linear operators!)

Independence: $X \perp Y \implies E[XY] = E[X]E[Y]$.

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Mean $\equiv E[X]$. An estimate of the distribution's location or central tendency.

Variance, $Var[X] \equiv E[(X - E[X])^2] = (\text{standard deviation})^2$. An estimate of spread.

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The expectation value $E[g(X)]$ of a function $g(X)$ of a random variable X, is the weighted average of $g(X)$, with the weights being the associated probabilities:

Discrete: $E[g(X)] = \sum_{i=1}^{N} g(x_i)\ P(X = x_i)$. Continuous: $E[g(X)] = \int_{t=-\infty}^{t=\infty} g(x)\ p_X(x)\ dx$.

Linearity: $E[\alpha g(X) + \beta h(X)] = \alpha E[g(X)] + \beta E[h(X)]$. ($\sum$ and $\int$ are linear operators!)

Independence: $X \perp Y \implies E[XY] = E[X]E[Y]$.

Mean $\equiv E[X]$. An estimate of the distribution's location or central tendency.

Variance, $Var[X] \equiv E[(X - E[X])^2] = (\text{standard deviation})^2$. An estimate of spread.

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The expression for the variance can be simplified: $Var[X] = E[X^2] - (E[X])^2$.

Note: When no other function is specified, "expectation value" refers to the mean, $E(X)$.

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3. For constants α, β and random variables X, Y,

   $Var(\alpha X + \beta Y) = ??$

   Evaluate this expression using the definition of variance in terms of expectation values.

Covariance

$$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y) + 2\alpha\beta \overbrace{E[(X - E[X])(Y - E[Y])]}^{\text{interpretation?}} \quad (1)$$

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Therefore, $Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y) + 2\alpha\beta Cov(X, Y)$.

If the two variables are uncorrelated, then the third term vanishes.

Correlation coefficient

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By definition, $\rho_{XX} = 1$. "Perfect correlation".

$\rho_{XX} = -1$: "perfect anticorrelation".

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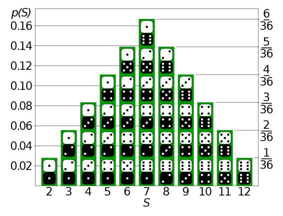

(Tim Stellmach/Public Domain)

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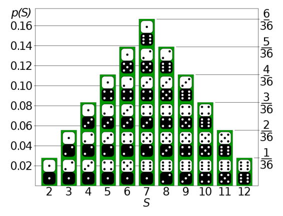

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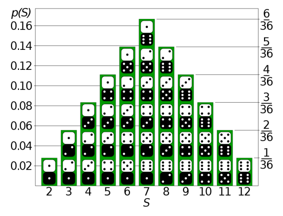

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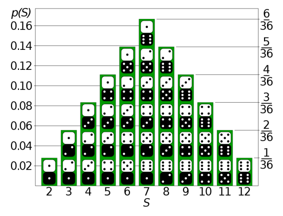

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As # measurements $N \uparrow$, variance on the mean of N measurements $\downarrow$.
Sample mean approaches population mean.

See Law of Large Numbers.

Some common probability distributions

Attributes of probability distributions via Python/Scipy

(See documentation for each distribution in `scipy.stats`)

- `rvs` - random variates (sample from the distribution)
- `pmf/pdf` - PMF or PDF
- `logpmf/logpdf` - log of the PMF or PDF
- `cdf` - CDF
- `logcdf` - log of the CDF
- `ppf` - percent point function (inverse of `cdf`; percentiles)
- `stats` - Mean('m'), variance('v'), skew('s'), kurtosis('k')
  (also see `mean`, `median`, `var`, `std`)
- `expect` - Compute expectation value of a function of this random variable
- `interval` - Confidence interval

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with $\mathbb{I}(x)$ the Indicator (or Heaviside) function: $\mathbb{I}_{x \in \{0,1\}}(x) = \begin{cases} 1, & \text{if } x \in \{0, 1\} \\ 0, & \text{otherwise} \end{cases}$

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Generate 10 samples from `scipy.stats.bernoulli`:

```
from scipy.stats import bernoulli
p = 0.25 #probability of success
print(bernoulli.rvs(p, size = 10)) #10 random deviates
[0 0 0 1 1 0 0 0 0 0] #possible output
```

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Compute probability of 2 successes in 10 trials

```
from scipy.stats import binom
n, k, p = 10, 2, 0.25 #total trials, num successes, prob of 1 success
print(binom.pmf(k, n, p)) #prob of k successes in n trials

0.28156757354736334 #output
```

Expectation value of a Binomial Distribution

Recall:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad (2)$$

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$$E(X) = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} \text{ (}k=0\text{ term vanishes)}$$

$$= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \text{using Eq. (3)}$$

Expectation value of a Binomial Distribution

Recall:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (2)$$

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&= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \text{using Eq. (3)} \\
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&= np \sum_{s=0}^{n-1} \binom{n-1}{s} p^s (1-p)^{n-1-s}
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\end{aligned}$$

Similarly, we can compute $Var[X]$ using $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$