Stellar Atmospheres: Lecture 3, 2020.04.20

Prof. Sundar Srinivasan

IRyA/UNAM

Mechanisms for (de)populating energy levels

Interaction with photons (radiative) or other atoms/ions (collisional).

Excitation: radiative (Einstein B coefficient) or collisional.

De-excitation: radiative (spontaneous – Einstein A coefficient/stimulated – Einstein B coefficient) or collisional.

Mechanisms for (de)populating energy levels

Interaction with photons (radiative) or other atoms/ions (collisional).

Excitation: radiative (Einstein B coefficient) or collisional.

De-excitation: radiative (spontaneous – Einstein A coefficient/stimulated – Einstein B coefficient) or collisional.

Elastic collisions (total KE is conserved) $\rightarrow$ Boltzmann Distribution (of velocities) $\rightarrow$ thermal equilibrium.

Collisional (de)excitations are examples of inelastic collisions: some of the initial KE is either used to excite atom to higher level, or is converted into radiation upon de-excitation.

High density $\rightarrow$ collisions dominate. Low density $\rightarrow$ (de)excitations disrupt thermal equilibrium.

Ionisation

Neutral ("HI") $\rightarrow$ ionised ("HII") hydrogen
$\Rightarrow$ # particles double at fixed T, p doubles, μ halves (electrons have no mass).

Ionisation

Neutral ("H I") $\rightarrow$ ionised ("H II") hydrogen
$\Rightarrow$ # particles double at fixed T, p doubles, μ halves (electrons have no mass).

Effect on spectra:

H I: discrete energy levels, produces line features in spectrum.
In general, neutral atoms and ions contribute to line opacity.

Ionisation

Neutral ("H I") $\rightarrow$ ionised ("H II") hydrogen
$\Rightarrow$ # particles double at fixed T, p doubles, μ halves (electrons have no mass).

Effect on spectra:

H I: discrete energy levels, produces line features in spectrum.
In general, neutral atoms and ions contribute to line opacity.

Ions also contribute to continuum opacity (ionisation and KE to freed electrons)
Proton: no absorption (except free-free with electron).
Hydride (H^-): electronic structure like He, with 1/4 the nuclear mass. Optical through NIR absorption.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

– Maximise the entropy subject to normalisation and fixed average energy.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

- Maximise the entropy subject to normalisation and fixed average energy.
- Form of the entropy function gives exponential dependence on energy.
  Multi-component systems (e.g., ions + electrons) must satisfy $P(E) = P(E_1) \cdot P(E_2) \cdots$
  Exponential dependence is the only possibility.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

- Maximise the entropy subject to normalisation and fixed average energy.
- Form of the entropy function gives exponential dependence on energy.
  Multi-component systems (e.g., ions + electrons) must satisfy $P(E) = P(E_1) \cdot P(E_2) \cdots$
  Exponential dependence is the only possibility.
- Average energy constraint $\rightarrow$ dependence is negative exp. with energy in units of kT.
- Normalisation constraint $\rightarrow$ partition function (next slide).

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

- Maximise the entropy subject to normalisation and fixed average energy.
- Form of the entropy function gives exponential dependence on energy.
  Multi-component systems (e.g., ions + electrons) must satisfy $P(E) = P(E_1) \cdot P(E_2) \cdots$
  Exponential dependence is the only possibility.
- Average energy constraint $\rightarrow$ dependence is negative exp. with energy in units of kT.
- Normalisation constraint $\rightarrow$ partition function (next slide).

states with energy E: depends on whether the accessible levels are

(1) discrete (*e.g.*, bound electronic states in H atom):
use $g(E)$, the degeneracy/multiplicity/occupation number for that energy.

(2) continuous (*e.g.*, free electrons):
estimate fraction of phase space volume occupied by states with energy E.

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

- Maximise the entropy subject to normalisation and fixed average energy.
- Form of the entropy function gives exponential dependence on energy.
  Multi-component systems (e.g., ions + electrons) must satisfy $P(E) = P(E_1) \cdot P(E_2) \cdots$
  Exponential dependence is the only possibility.
- Average energy constraint $\rightarrow$ dependence is negative exp. with energy in units of kT.
- Normalisation constraint $\rightarrow$ partition function (next slide).

states with energy E: depends on whether the accessible levels are

(1) discrete (*e.g.*, bound electronic states in H atom):
use $g(E)$, the degeneracy/multiplicity/occupation number for that energy.

(2) continuous (*e.g.*, free electrons):
estimate fraction of phase space volume occupied by states with energy E.
Typically, the differential volume of phase space

$$= \frac{d^3x\ d^3p}{h^3} = \frac{dV \times 4\pi p^2 dp}{h^3} = \frac{m^3 dV \times 4\pi v^2 dv}{h^3}.$$

Excitation and ionisation

Need thermal-equilibrium distribution of atoms among various excitation/ionisation states.

$P(\text{system in state with energy } E) \propto$ prob. of having energy E $\times$ # states with energy E.

Determining the function $P(E)$:

- Maximise the entropy subject to normalisation and fixed average energy.
- Form of the entropy function gives exponential dependence on energy.
  Multi-component systems (e.g., ions + electrons) must satisfy $P(E) = P(E_1) \cdot P(E_2) \cdots$
  Exponential dependence is the only possibility.
- Average energy constraint $\rightarrow$ dependence is negative exp. with energy in units of kT.
- Normalisation constraint $\rightarrow$ partition function (next slide).

states with energy E: depends on whether the accessible levels are

(1) discrete (*e.g.*, bound electronic states in H atom):
use $g(E)$, the degeneracy/multiplicity/occupation number for that energy.

(2) continuous (*e.g.*, free electrons):
estimate fraction of phase space volume occupied by states with energy E.
Typically, the differential volume of phase space

$$= \frac{d^3x\ d^3p}{h^3} = \frac{dV \times 4\pi p^2 dp}{h^3} = \frac{m^3 dV \times 4\pi v^2 dv}{h^3}.$$

Excitation: bound-bound transitions $\Rightarrow$ Boltzmann Distribution.
Ionisation: bound-free transitions $\Rightarrow$ Saha Equation.

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

For the Bohr model ($g_m = 2m^2, E_m \propto -1/m^2$), Z, summed over $m = 1$ to ∞, diverges!!

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

For the Bohr model ($g_m = 2m^2, E_m \propto -1/m^2$), Z, summed over $m = 1$ to ∞, diverges!! This is because an infinite volume is assumed for the single atom.

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

For the Bohr model ($g_m = 2m^2, E_m \propto -1/m^2$), Z, summed over $m = 1$ to ∞, diverges!! This is because an infinite volume is assumed for the single atom.

In practice, the higher levels are depopulated via collisions and interactions with charged species (low Debye Length).

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

For the Bohr model ($g_m = 2m^2, E_m \propto -1/m^2$), Z, summed over $m = 1$ to ∞, diverges!! This is because an infinite volume is assumed for the single atom.

In practice, the higher levels are depopulated via collisions and interactions with charged species (low Debye Length).
Solar core: $\lambda_D \sim$ Bohr radius, $n \sim 10^{31}$ m^{-3}, – electrons basically unbound.
Solar surface: $n \sim 10^6$ m^{-3}, $m_{\max} \sim 10^4$.

The partition function

The distribution of prob. for states in thermal equilibrium at temperature T is the Boltzmann Distribution:

$$P(E_i) = \frac{g_i}{Z} \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } P(E)dE = \frac{g(E)}{Z} \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous),}$$

Here, Z is a constant of normalisation called the partition function:

$$Z = \sum_i g_i \exp\left[-\frac{E_i}{kT}\right] \text{ (discrete), or } Z = \int g(E) \exp\left[-\frac{E}{kT}\right] dE \text{ (continuous)}$$

We will represent Z with $u(T)$ to explicitly note its dependence on temperature.

For the Bohr model ($g_m = 2m^2, E_m \propto -1/m^2$), Z, summed over $m = 1$ to ∞, diverges!! This is because an infinite volume is assumed for the single atom.

In practice, the higher levels are depopulated via collisions and interactions with charged species (low Debye Length).
Solar core: $\lambda_D \sim$ Bohr radius, $n \sim 10^{31}$ m^{-3}, – electrons basically unbound.
Solar surface: $n \sim 10^6$ m^{-3}, $m_{\max} \sim 10^4$.
Increasing densities in general will make it easier to free electrons from their orbits – reduction in ionisation potential. This pressure ionisation is common in stellar cores.

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

Consider the general case:

Initial: atom in ionisation state i, excitation state j.

Final: atom in ionisation state $i+1$, excitation state m, and

free electron with $KE = \dfrac{p^2}{2m_e}$.

Total energy change: $\Delta E = IP_{i+1} + E_{i+1,m} - E_{i,j} + KE$.

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

Consider the general case:

Initial: atom in ionisation state i, excitation state j.

Final: atom in ionisation state $i+1$, excitation state m, and

free electron with $KE = \dfrac{p^2}{2m_e}$.

Total energy change: $\Delta E = IP_{i+1} + E_{i+1,m} - E_{i,j} + KE$.

$g_{\text{initial}} = g_{i,j}$. Single value since discrete energy level.
After ionisation, atom is still in discrete energy state but electron isn't.

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

Consider the general case:

Initial: atom in ionisation state i, excitation state j.

Final: atom in ionisation state $i+1$, excitation state m, and

free electron with $KE = \dfrac{p^2}{2m_e}$.

Total energy change: $\Delta E = IP_{i+1} + E_{i+1,m} - E_{i,j} + KE$.

$g_{\rm initial} = g_{i,j}$. Single value since discrete energy level.
After ionisation, atom is still in discrete energy state but electron isn't.

$g_{\rm final} = g_{i+1,m} \cdot g_e$, where g_e is now the density of states: $g_e = \dfrac{8V\pi p^2 dp}{h^3} = \dfrac{8\pi p^2 dp}{n_e h^3}$.

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

Consider the general case:

Initial: atom in ionisation state i, excitation state j.

Final: atom in ionisation state $i+1$, excitation state m, and

free electron with $KE = \dfrac{p^2}{2m_e}$.

Total energy change: $\Delta E = IP_{i+1} + E_{i+1,m} - E_{i,j} + KE$.

$g_{\mathrm{initial}} = g_{i,j}$. Single value since discrete energy level.
After ionisation, atom is still in discrete energy state but electron isn't.

$g_{\mathrm{final}} = g_{i+1,m} \cdot g_e$, where g_e is now the density of states: $g_e = \dfrac{8V\pi p^2 dp}{h^3} = \dfrac{8\pi p^2 dp}{n_e h^3}$.

$$\Rightarrow \frac{dN_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi p^2}{n_e h^3} \exp\left[-\frac{1}{2}\frac{p^2}{m_e kT}\right] dp,$$

The Saha Equation - I

Ionisation rate $\propto IP_i$ (ionisation potential) and T. Recombination rate $\propto T$ and n_e (or p_e).

Consider the general case:

Initial: atom in ionisation state i, excitation state j.
Final: atom in ionisation state $i+1$, excitation state m, and

free electron with $KE = \dfrac{p^2}{2m_e}$.

Total energy change: $\Delta E = IP_{i+1} + E_{i+1,m} - E_{i,j} + KE$.

$g_{\text{initial}} = g_{i,j}$. Single value since discrete energy level.
After ionisation, atom is still in discrete energy state but electron isn't.

$g_{\text{final}} = g_{i+1,m} \cdot g_e$, where g_e is now the density of states: $g_e = \dfrac{8V\pi p^2 dp}{h^3} = \dfrac{8\pi p^2 dp}{n_e h^3}$.

$$\Rightarrow \frac{dN_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi p^2}{n_e h^3} \exp\left[-\frac{1}{2}\frac{p^2}{m_e kT}\right] dp,$$

where $dN_{i+1,m}$ is the number of atoms in ionisation state $i+1$ and excitation state m that have lost electrons with momenta in the range $p, p+dp$.

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \dfrac{p}{\sqrt{2m_e kT}}$, and note that $\int\limits_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \dfrac{\sqrt{\pi}}{4}$:

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int\limits_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3} (2m_e kT)^{3/2} \frac{\sqrt{\pi}}{4}$$

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3} (2m_e kT)^{3/2} \frac{\sqrt{\pi}}{4}$$

$$= \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3} (2m_e kT)^{3/2} \frac{\sqrt{\pi}}{4}$$

$$= \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$

Simplify: sum over all possible excitation states, get the relative populations of successive ionisation states regardless of the excitation state.

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3}(2m_e kT)^{3/2} \frac{\sqrt{\pi}}{4}$$
$$= \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$

Simplify: sum over all possible excitation states, get the relative populations of successive ionisation states regardless of the excitation state.

Note: $\sum_{m=1}^{\infty} N_{i,j} \propto \sum_{m=1}^{\infty} g_{i,j} \exp\left[-\frac{E_{i,j}}{kT}\right] \equiv u_i(T)$, the partition function for ionisation state i.

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int\limits_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3}(2m_e kT)^{3/2}\frac{\sqrt{\pi}}{4}$$

$$= \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$

Simplify: sum over all possible excitation states, get the relative populations of successive ionisation states regardless of the excitation state.

Note: $\sum\limits_{m=1}^{\infty} N_{i,j} \propto \sum\limits_{m=1}^{\infty} g_{i,j} \exp\left[-\frac{E_{i,j}}{kT}\right] \equiv u_i(T)$, the partition function for ionisation state i.

$$\frac{N_{i+1}\ n_e}{N_i} = 2\frac{u_{i+1}(T)}{u_i(T)} \exp\left[-\frac{IP_{i+1}}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$ (we also moved n_e to the LHS).

The Saha Equation - II

To integrate over the momenta, we set $\zeta = \frac{p}{\sqrt{2m_e kT}}$, and note that $\int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}$:

$$\frac{N_{i+1,m}}{N_{i,j}} = \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \frac{4\pi}{h^3}(2m_e kT)^{3/2} \frac{\sqrt{\pi}}{4}$$

$$= \frac{2g_{i+1,m}}{g_{i,j}\ n_e} \exp\left[-\frac{IP_{i+1}}{kT}\right] \exp\left[-\frac{(E_{i+1,m} - E_{i,j})}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$

Simplify: sum over all possible excitation states, get the relative populations of successive ionisation states regardless of the excitation state.

Note: $\sum_{m=1}^{\infty} N_{i,j} \propto \sum_{m=1}^{\infty} g_{i,j} \exp\left[-\frac{E_{i,j}}{kT}\right] \equiv u_i(T)$, the partition function for ionisation state i.

$$\frac{N_{i+1}\ n_e}{N_i} = 2\frac{u_{i+1}(T)}{u_i(T)} \exp\left[-\frac{IP_{i+1}}{kT}\right] \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}$$ (we also moved n_e to the LHS).

Saha Equation in terms of the final electron density n_e. Use $p_e = n_e kT$ to get pressure version:

$$\frac{N_{i+1}\ p_e}{N_i} = 2\frac{u_{i+1}(T)}{u_i(T)} \exp\left[-\frac{IP_{i+1}}{kT}\right] \left(\frac{2\pi m_e}{h^2}\right)^{3/2} (kT)^{5/2}$$

Application: the Balmer Thermometer

Apply the Saha equation to H I and H II:

$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

Application: the Balmer Thermometer

Apply the Saha equation to HI and HII:

$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

$$\frac{N_{\mathrm{HII}}\ n_e}{N_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{IP_1}{kT}\right].$$

Application: the Balmer Thermometer

Apply the Saha equation to HI and HII:

$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

$$\frac{N_{\mathrm{HII}}\ n_e}{N_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{IP_1}{kT}\right].$$

For $n_e = 10^{20}$ m^{-3}, this ratio ~ 1 for $T = 10^4$ K.
Beyond this, fewer atoms are available for excitation.

Application: the Balmer Thermometer

Apply the Saha equation to HI and HII:

$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

$$\frac{N_{\mathrm{HII}}\ n_e}{N_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{IP_1}{kT}\right].$$

For $n_e = 10^{20}$ m^{-3}, this ratio ~ 1 for $T = 10^4$ K. Beyond this, fewer atoms are available for excitation.

At very low T: very few atoms excitable. $T \gtrsim 5000$ K required for Balmer lines.

Application: the Balmer Thermometer

Apply the Saha equation to HI and HII:

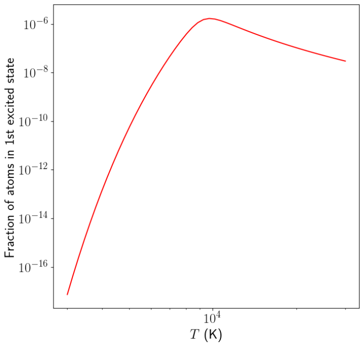
$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

$$\frac{N_{\mathrm{HII}}\ n_e}{N_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{IP_1}{kT}\right].$$

For $n_e = 10^{20}$ m^{-3}, this ratio ~ 1 for $T = 10^4$ K. Beyond this, fewer atoms are available for excitation.

At very low T: very few atoms excitable. $T \gtrsim 5000$ K required for Balmer lines.

At very high T: most atoms ionised. $T \lesssim 20000$ K required for Balmer lines.



Application: the Balmer Thermometer

Apply the Saha equation to HI and HII:

$g_{\mathrm{HI},1} = 2$, and g_{HII} (just a proton). Ratio of partition functions is then 1/2, and

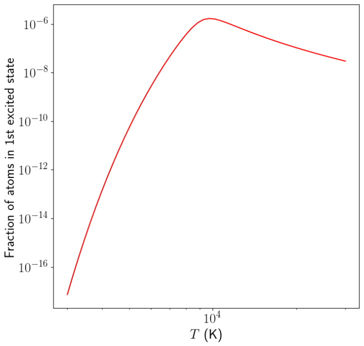
$$\frac{N_{\mathrm{HII}}\ n_e}{N_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{IP_1}{kT}\right].$$

For $n_e = 10^{20}\ \mathrm{m}^{-3}$, this ratio ~ 1 for $T = 10^4$ K. Beyond this, fewer atoms are available for excitation.

At very low T: very few atoms excitable. $T \gtrsim 5000$ K required for Balmer lines.

At very high T: most atoms ionised. $T \lesssim 20000$ K required for Balmer lines.

Balmer lines have peak intensity around $T = 10^4$K.



Spectroscopic classification of stars – I

Elements of a classification scheme

(Keenan, from *Stars and Stellar Systems III: Basic Astronomical Data* 3rd ed., 1969)

Spectroscopic classification of stars – I

Elements of a classification scheme

1. **Selection criteria** – the features (relative intensities of lines/bands, etc.) used in ordering the spectra

(Keenan, from *Stars and Stellar Systems III: Basic Astronomical Data* 3rd ed., 1969)

Spectroscopic classification of stars – I

Elements of a classification scheme

1. **Selection criteria** – the features (relative intensities of lines/bands, etc.) used in ordering the spectra
2. **Standard stars** – order spectra using the above criteria, arrange the spectra using numbers/symbols to describe the positions of stars in the sequence, then identify good representative stars for each type
3. **Calibration** – connect the spectral types to physical characteristics of the stars

(Keenan, from *Stars and Stellar Systems III: Basic Astronomical Data* 3rd ed., 1969)

Spectroscopic classification of stars – I

Elements of a classification scheme

1. **Selection criteria** – the features (relative intensities of lines/bands, etc.) used in ordering the spectra
2. **Standard stars** – order spectra using the above criteria, arrange the spectra using numbers/symbols to describe the positions of stars in the sequence, then identify good representative stars for each type
3. **Calibration** – connect the spectral types to physical characteristics of the stars

(Keenan, from *Stars and Stellar Systems III: Basic Astronomical Data* 3rd ed., 1969)

Originally, classification was done on a completely empirical basis – organised by intensity of hydrogen lines w.r.t. other lines. Reordered when relation to physical parameters became clearer.

Spectroscopic classification of stars – II

Originally assigned alphabet designations after organising by strength of H lines.

Spectroscopic classification of stars – II

Originally assigned alphabet designations after organising by strength of H lines.
Rearranged once it became clear that line strengths increased then decreased with temperature.

Spectroscopic classification of stars – II

Originally assigned alphabet designations after organising by strength of H lines.
Rearranged once it became clear that line strengths increased then decreased with temperature.
Temperature sequence:

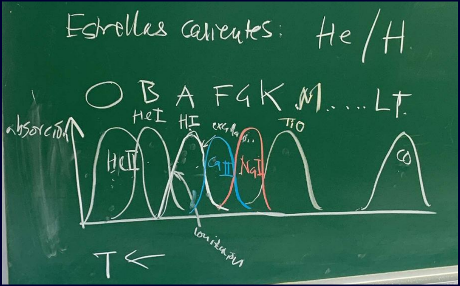
```
            C[R/N]
O B A F G K M      L T
              S
```

Spectroscopic classification of stars – II

Originally assigned alphabet designations after organising by strength of H lines.
Rearranged once it became clear that line strengths increased then decreased with temperature.

Temperature sequence:

C[R/N]
O B A F G K M L T
S



credit: W. Henney

Table 1.1. *Spectral classification.*

Temperature criteria	
O4–B0	He I λ4471/He II λ4541 increasing with type He I + He II λ4026/He II λ4200 increasing with type
B0–A0	H lines increase with type He I lines reach maximum at B2 Ca II K line becomes visible at B8
A0–F5	Ca II K/H$_\delta$ increasing with type Neutral metals become stronger *G* band visible starting at F2
F5–K2	H lines decrease Neutral metals increase *G* band strengthens
K2–M5	*G* band changes appearance Ca I λ4226 increases rapidly TiO starts near M0 in dwarfs, K5 in giants
Luminosity criteria	
O9–A5	H and He lines weaken with increasing luminosity Fe II becomes prominent in A0–A5
F0–K0	Blend $\lambda\lambda$4172–9 increases with luminosity (early F) Sr II λ4077 increases with luminosity CN λ4200 increases with luminosity
K0–M6	CN λ4200 increases with luminosity Sr II increases with luminosity
Suffix notation	
e	Emission lines are present
f	He II λ4686 and/or C III λ4650 in emission; mostly for O stars
k	Ca II K line when unexpected, e.g., interstellar in hot stars
m	Metallic; metal lines are stronger than normal
n	Nebulous; lines are broad and shallow; usually high rotation
nn	Very nebulous!
p	Peculiar; spectrum is abnormal
q	Queer; unusual emission; evolved from Q novae designation (archaic)
s	Sharp; lines are sharp, usually for early-type stars with low rotation
v	Variable; spectrum changes with time
w	Wolf–Rayet bands present (archaic)
Old prefix notation	
c for supergiants; g for giants; d for dwarfs	

from Ch. 1 of Gray

Spectroscopic classification of stars – III

Luminosity class

In solar units, $L = R^2 T_{\rm eff}^4$.

$T_{\rm eff}$ known from spectral line ratios $\Rightarrow L \propto R^2$.

$\Rightarrow$ luminosity class = size class.

(Wikipedia/
https://www.cfa.harvard.edu/~pberlind/atlas/htmls/note.html)

Spectroscopic classification of stars – III

Luminosity class

In solar units, $L = R^2 T_{\rm eff}^4$.

$T_{\rm eff}$ known from spectral line ratios $\Rightarrow L \propto R^2$.

$\Rightarrow$ luminosity class = size class.

In terms of surface gravity, $L \propto R^2 \propto Mg^{-1}$.

(Wikipedia/
https://www.cfa.harvard.edu/∼pberlind/atlas/htmls/note.html)

Spectroscopic classification of stars – III

Luminosity class

In solar units, $L = R^2 T_{\rm eff}^4$.

$T_{\rm eff}$ known from spectral line ratios $\Rightarrow L \propto R^2$.

$\Rightarrow$ luminosity class = size class.

In terms of surface gravity, $L \propto R^2 \propto Mg^{-1}$.

Use mass-luminosity relation:

$L \propto M^{-\alpha}, \alpha \approx 3 - 3.5$ for "normal" stars (no red giants!)

(Wikipedia/

https://www.cfa.harvard.edu/∼pberlind/atlas/htmls/note.html)

Spectroscopic classification of stars – III

Luminosity class

In solar units, $L = R^2 T_{\rm eff}^4$.

$T_{\rm eff}$ known from spectral line ratios $\Rightarrow L \propto R^2$.

$\Rightarrow$ luminosity class = size class.

In terms of surface gravity, $L \propto R^2 \propto M g^{-1}$.

Use mass-luminosity relation:

$L \propto M^{-\alpha}, \alpha \approx 3 - 3.5$ for "normal" stars (no red giants!)

$$\Rightarrow L \propto g^{\dfrac{\alpha}{1-\alpha}},$$

and thus related to pressure (hydrostatic equilibrium).

Pressure classification = luminosity classification.

(Wikipedia/
https://www.cfa.harvard.edu/~pberlind/atlas/htmls/note.html)

Spectroscopic classification of stars – III

Luminosity class

In solar units, $L = R^2 T_{\rm eff}^4$.

$T_{\rm eff}$ known from spectral line ratios $\Rightarrow L \propto R^2$.

$\Rightarrow$ luminosity class = size class.

In terms of surface gravity, $L \propto R^2 \propto M g^{-1}$.

Use mass-luminosity relation:

$L \propto M^{-\alpha}, \alpha \approx 3 - 3.5$ for "normal" stars (no red giants!)

$$\Rightarrow L \propto g^{\frac{\alpha}{1-\alpha}},$$

and thus related to pressure (hydrostatic equilibrium).

Pressure classification = luminosity classification.

Yerkes luminosity classes

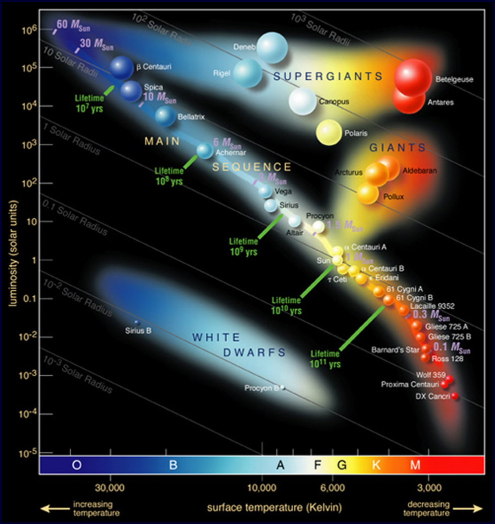
Luminosity class	Description	Examples
0 *or* Ia+	hypergiants or extremely luminous supergiants	Cygnus OB2#12 – B3-4Ia+ [18]
Ia	luminous supergiants	Eta Canis Majoris – B5Ia [19]
Iab	intermediate-size luminous supergiants	Gamma Cygni – F8Iab [20]
Ib	less luminous supergiants	Zeta Persei – B1Ib [21]
II	bright giants	Beta Leporis – G0II [22]
III	normal giants	Arcturus – K0III [23]
IV	subgiants	Gamma Cassiopeiae – B0.5IVpe [24]
V	main-sequence stars (dwarfs)	Achernar – B6Vep [21]
sd (*prefix*) *or* VI	subdwarfs	HD 149382 – sdB5 *or* B5VI [25]
D (*prefix*) *or* VII	white dwarfs [c]	van Maanen 2 – DZ8 [26]

(Wikipedia/
https://www.cfa.harvard.edu/∼pberlind/atlas/htmls/note.html)

The Hertzsprung-Russell Diagram

A plot of L vs $T_{\rm eff}$.

Observational version: $L \rightarrow$ (absolute) magnitude, $T_{\rm eff} \rightarrow$ colour.



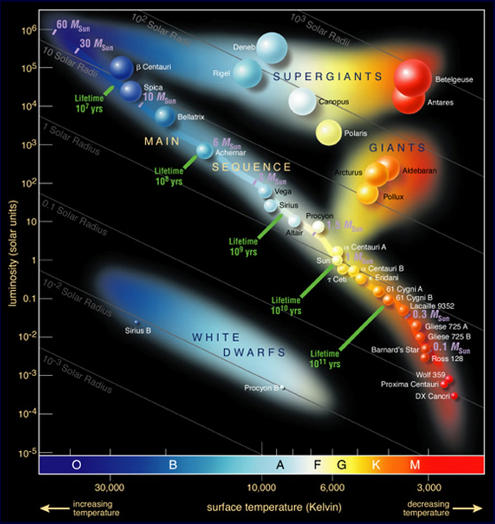
Credit: Bennett et al., “The Cosmic Perspective”, 8 ed.

The Hertzsprung-Russell Diagram

A plot of L vs $T_{\rm eff}$.

Observational version: $L \rightarrow$ (absolute) magnitude, $T_{\rm eff} \rightarrow$ colour.

Important features:



Credit: Bennett et al., “The Cosmic Perspective”, 8 ed.

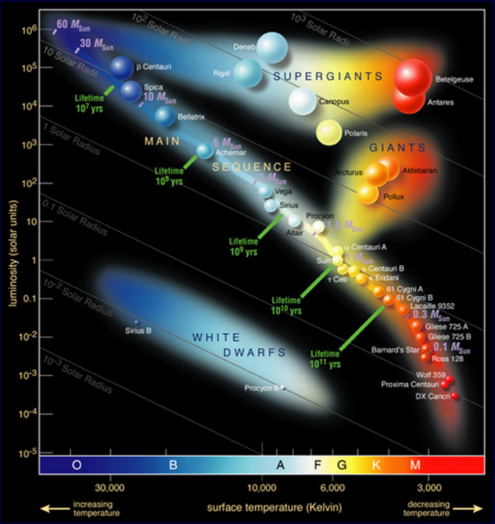
The Hertzsprung-Russell Diagram

A plot of L vs $T_{\rm eff}$.

Observational version: $L \rightarrow$ (absolute) magnitude, $T_{\rm eff} \rightarrow$ colour.

Important features:

Main sequence – hydrogen-burning cores.



Credit: Bennett et al., "The Cosmic Perspective", 8 ed.

The Hertzsprung-Russell Diagram

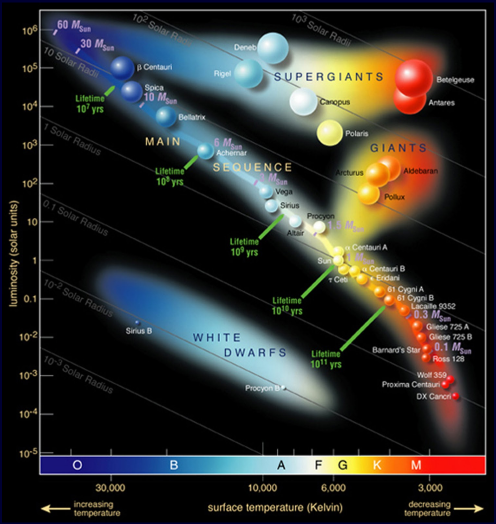
A plot of L vs $T_{\rm eff}$.

Observational version: $L \rightarrow$ (absolute) magnitude, $T_{\rm eff} \rightarrow$ colour.

Important features:

Main sequence – hydrogen-burning cores.

Red and asymptotic giant branch – shell hydrogen- and helium-burning.



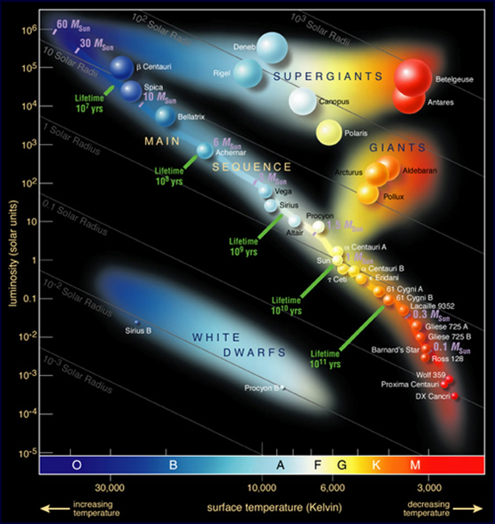
Credit: Bennett et al., "The Cosmic Perspective", 8 ed.

The Hertzsprung-Russell Diagram

A plot of L vs T_{eff}.

Observational version: $L \rightarrow$ (absolute) magnitude, $T_{\mathrm{eff}} \rightarrow$ colour.

Important features:
Main sequence – hydrogen-burning cores.
Red and asymptotic giant branch – shell hydrogen- and helium-burning.
Supergiants – massive, short-lived.



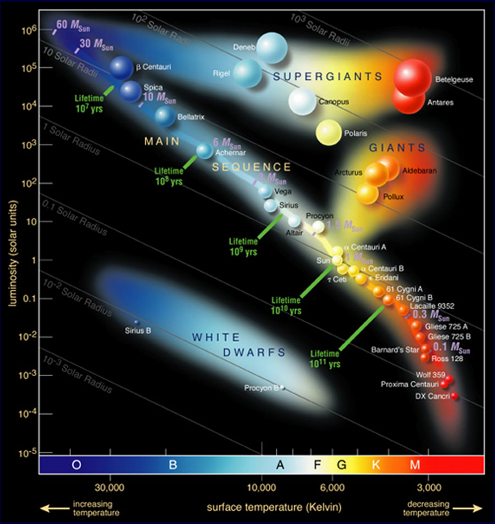
Credit: Bennett et al., "The Cosmic Perspective", 8 ed.

The Hertzsprung-Russell Diagram

A plot of L vs $T_{\rm eff}$.

Observational version: $L \to$ (absolute) magnitude, $T_{\rm eff} \to$ colour.

Important features:
Main sequence – hydrogen-burning cores.
Red and asymptotic giant branch – shell hydrogen- and helium-burning.
Supergiants – massive, short-lived.
White dwarfs – end products of giants.



Credit: Bennett et al., "The Cosmic Perspective", 8 ed.