# Stellar Atmospheres: Lecture 05, 2020.04.29

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Moments of intensity w.r.t. $\mu$. Weighted means of $I_\nu(\mu)$ with weights 1, $\mu$, $\mu^2$:

$$I_\nu^{(0)} = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu) d\mu \equiv J_\nu; \qquad I_\nu^{(1)} = \frac{1}{2}\int_{-1}^{1} \mu\, I_\nu(\mu) d\mu \equiv H_\nu; \qquad I_\nu^{(2)} = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu(\mu) d\mu \equiv K_\nu.$$

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Since the integral over solid angle is $2\pi$ times the integral over $\mu$, we can relate each of these moments to physical quantities:

$\dfrac{4\pi J_\nu}{c} = u_\nu =$ energy density (scalar);

$H_\nu = 4\pi F_\nu$, where $F_\nu$ is the net flux along **n** (component of the flux vector);

$\dfrac{4\pi K_\nu}{c} = p_\nu$, the momentum flux along **n** (component of the rank-2 pressure tensor).

# References

Rybicki & Lightman, chapter 1.

S. R. Cranmer, UC Boulder, course notes.

S. Owocki, U Delware, course notes.

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   $$I_\nu(\mu) = \begin{cases} I_+, & \text{if } \mu > 0 \\ I_-, & \text{otherwise} \end{cases}$$

   For the stellar atmosphere, there is a nonzero net flux along $\mu > 0$, so $I_+ > I_-$.

   For this case, $J_\nu = \frac{1}{2}(I_+ + I_-)$; $H_\nu = \frac{1}{2}(I_+ - I_-)$; $K_\nu = \frac{1}{6}(I_+ + I_-) = \frac{1}{3}J_\nu$.

   Note: If $\frac{I_-}{I_+} \ll 1$, then $H \approx \frac{J}{2}$.

# Elastic scattering

The scattering of light by atoms/molecules/ions/electrons can be elastic or inelastic. In elastic scattering, the energy (frequency) of the photon remains the same as it was before it was scattered. The direction may, in general, change.
Some examples of elastic scattering:

1. Thomson Scattering: electric field of incident photon accelerates a charged particle of mass $m$. Scattering is elastic in the low-energy (non-relativistic) regime, $h\nu \ll mc^2$. This is equivalent to requiring $\lambda \gg \lambda_{\rm Compton}$, where $\lambda_{\rm Compton} = \dfrac{h}{mc}$ is the Compton wavelength of the particle.
2. Rayleigh Scattering: particle sizes are much smaller than the wavelength: $\lambda \gg a$. Used to explain scattering of sunlight by the Earth's atmosphere. Scattered intensity is
   $$I(\lambda, \mu) \propto \frac{1+\mu^2}{\lambda^4}, \qquad \text{where } \mu = \cos(\text{scattering angle})$$
   The wavelength dependence implies that blue light is scattered to a much higher extent than red light, explaining the red colour of sunsets.
3. Mie Scattering: scattering by spherical particles with sizes comparable to the wavelength: $\lambda \approx a$. Standard approximation used to account for the effects of scattering from circumstellar and interstellar dust grains.

# Source function for absorption and scattering

Pure absorption:

The emission from any material is always $\leq B_\nu(T)$, and is maximum at thermal equilibrium.

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Simplifications: scattering is assumed to be
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2) coherent/elastic/monochromatic – # of photons conserved in every frequency range.

$\Rightarrow$ for every $d\nu$, power absorbed = power emitted $\Rightarrow \underbrace{j_\nu = \alpha_\nu^{\rm sca} J_\nu}_{\text{monochromatic radiative equilibrium}} \Rightarrow S_\nu^{\rm sca} = J_\nu$.

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Absorption + scattering = attenuation or extinction: $\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca} = \alpha_\nu^{\rm ext}$.

RT equation for absorption + (isotropic + coherent) scattering: $\mu \frac{dI_\nu}{dz} = -\alpha_\nu^{\rm ext} (I_\nu - S_\nu)$.

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$$S_\nu = \frac{j_\nu}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}} = (1 - a_\nu) B_\nu(T) + a_\nu J_\nu, \text{ where } a_\nu = \frac{\alpha_\nu^{\rm sca}}{\alpha_\nu^{\rm ext}} = \text{scatt. albedo}.$$

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Assumptions of thermal equilibrium and isotropic scattering $\Rightarrow S_\nu$ is isotropic.
Complication: $S_\nu$ depends on $J_\nu$ which depends on $I_\nu$.
Simplification: LTE approximation, Grey approximation, and the Eddington Approximation.

# Grey atmospheres

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Let us write $d\tau \equiv -\alpha^{\rm ext} dz$ negative sign to show that $\tau \uparrow$ as $z \downarrow$ from surface into the interior.

Two equations for grey, LTE, plane-parallel (GLPP) atmospheres:

$$\mu \frac{dI}{d\tau} = I - (1-a)B - aJ$$ (radiative transfer)

$$\frac{dF}{d\tau} = 0$$ (energy cons./rad. equilibrium)

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Now compute moments of the RT equation to relate to physical quantities ($u$, $F$, $p$).

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$\Rightarrow K = H\tau + K(\tau = 0)$ ($\tau = 0$ represents the surface).

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Moment closure: need another relation to terminate the series.

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$$\Rightarrow J(\tau) = 3H\left(\tau + \frac{2}{3}\right)$$

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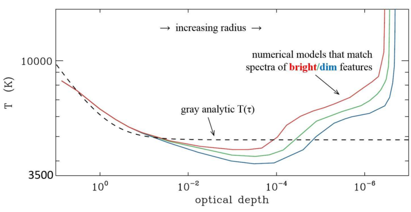
Temperature stratification in GLPP + Edd. atmosphere:

$$T^4 = \frac{3}{4}T_{\rm eff}^4\left(\tau + \frac{2}{3}\right)$$




Credit: S. R. Cranmer, UC Boulder

# GLPP + Edd. Application #1: Temperature Stratification

In radiative equilibrium, $S = B = J$.

Use $J(\tau) = 3H\left(\tau + \frac{2}{3}\right)$ and the Stefan-Boltzmann Law, $B(T) = \frac{\sigma}{\pi}T^4$:

$$\frac{\sigma}{\pi}T^4 = 3H\left(\tau + \frac{2}{3}\right)$$

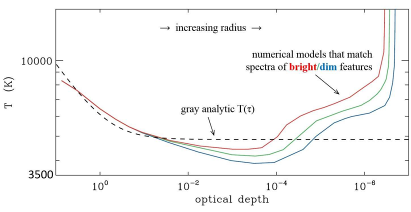
Also, $F = 4\pi H$, and $F = \sigma T_{\rm eff}^4$ (this equation defines $T_{\rm eff}$) $\Longrightarrow$

Temperature stratification
in GLPP + Edd. atmosphere:

$$T^4 = \frac{3}{4}T_{\rm eff}^4\left(\tau + \frac{2}{3}\right)$$

Optical photosphere: $T(\tau = 2/3) = T_{\rm eff}$.

$T(\tau = 0) = 0.841\,T_{\rm eff}$ (actual min. $\sim 0.7\,T_{\rm eff}$).

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# Formal solution of RT eq. for GLPP + Edd. atmosphere

Using an integrating factor as before, we get

$$I(\tau, \mu) = \exp\left[\frac{\tau}{\mu}\right] \int_{\tau_1}^{\tau_2} \frac{dt}{\mu} S(t) \exp\left[-\frac{t}{\mu}\right], \qquad \text{where } (\tau_1, \tau_2) = \begin{cases} (\tau, \infty), & \mu > 0 \text{(upward ray)} \\ (0, \tau), & \mu < 0 \text{(downward ray)} \end{cases}$$

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**Full solution for GLPP + Edd. atmosphere**

$$I_\nu(\tau,\mu) = \begin{cases} 3H\left(\tau + \mu + \frac{2}{3}\right) & \mu > 0 \text{(upward ray)} \\ 3H\left(\tau + \mu + \frac{2}{3} - \left(\mu + \frac{2}{3}\right)\exp\left[-\frac{\tau}{\mu}\right]\right), & \mu < 0 \text{(downward ray)} \end{cases}$$

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This approximate relation is accurate over a large range, because the exponential 'localises' the source function to a range around $\tau = \mu$.

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E-B Relation: $S(\tau) = S(\tau = 0) + \tau \left(\frac{dS}{d\tau}\right)_{\tau=0}$, with $S(\tau = 0) = 2H$, $\left(\frac{dS}{d\tau}\right)_{\tau=0} = 3H$

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Credit: Brocken Inaglory, CC BY 2.5, via Wikimedia Commons.

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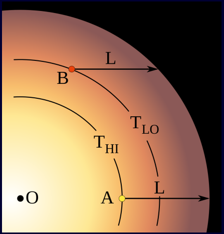
Prediction easily confirmed by observations (please do not stare directly at the Sun).

# Limb Darkening: Qualitative Explanation




Credit: User:Prboks13/Wikimedia, Public Domain

Different sight lines to the star are provided by emergent rays at differing $\mu$ values.

As projected distance from stellar centre increases, $\mu$ decreases.

Photons observed at lower $\mu$ arise from shallower regions (lower depth) than those at higher $\mu$, and thus correspond to a cooler temperature, and therefore a lower intensity.

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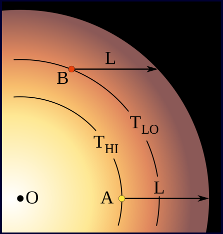


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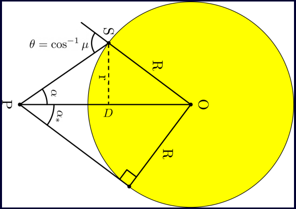
It is also possible to have limb brightening in optically thin regions where the temperature decreases with depth (e.g., the Solar corona). It could also arise at wavelengths corresponding to optically thin line emission from certain species.

# Limb Darkening as a function of projected distance

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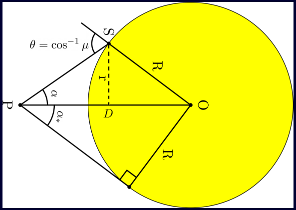


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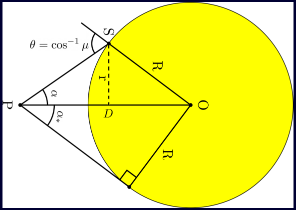




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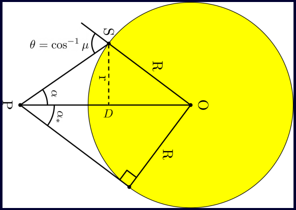




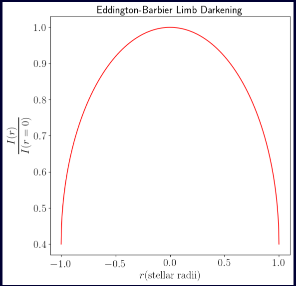


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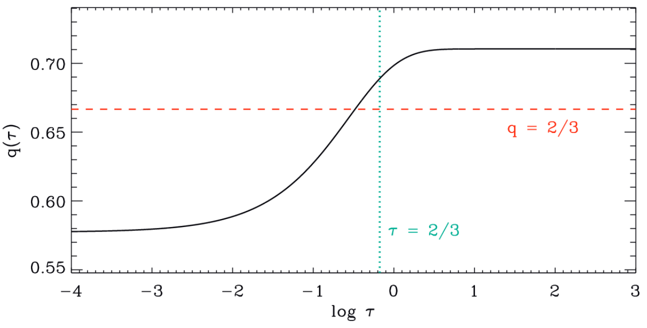


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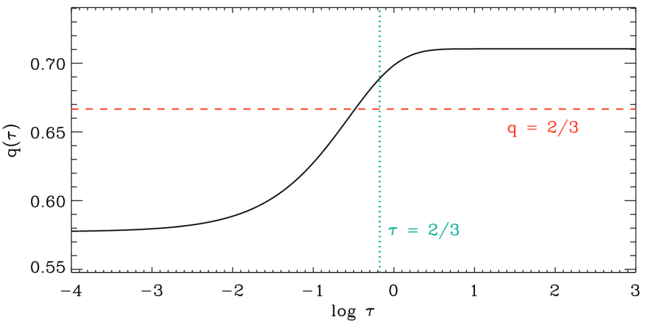


Credit: S. R. Cranmer, UC Boulder

Hopf's analysis gives an exact value for $q$ at the surface: $q(\tau = 0) = \frac{1}{\sqrt{3}}$ (instead of $\frac{2}{3} \approx 0.67$).