Stellar Atmospheres: Lecture 9, 2020.05.21

Prof. Sundar Srinivasan

IRyA/UNAM

References

1. Collins, *The Fundamentals of Stellar Astrophysics*, Ch 13, 14.
2. Hubeny & Mihalas, Ch. 8, 17
3. Böhm-Vitense, Ch 10.

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Type of Perturber	*Degenerate*	*Nondegenerate*
Ion or electron	Linear Stark effect $n = 2$	Quadratic Stark effect $n = 4$
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- Applies to non-degenerate levels (energy levels well separated).
  - Reasonable models for electron collisions.
  - Emission "interrupted" by near-instantaneous collision $\rightarrow$ phase shift/transition in line or to another atomic level.
  - Start/stop $\Longrightarrow$ frequency spread and shift of line centre.
  - Resulting profile still Lorentzian, but with modified Γ!
  - Lifetime of state reduced: $A_{ij} \longrightarrow A_{ij} + Y_e$, where $Y_e = n_e \left\langle \sigma(v_e) v_e \right\rangle$ is the collision rate.
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$t_{\rm coll} \gg t_{\rm rad} \rightarrow$ largest effect on the wings (broadening) $\rightarrow$ quasi-static approximation.

Required to treat degenerate levels (energy separation very small).

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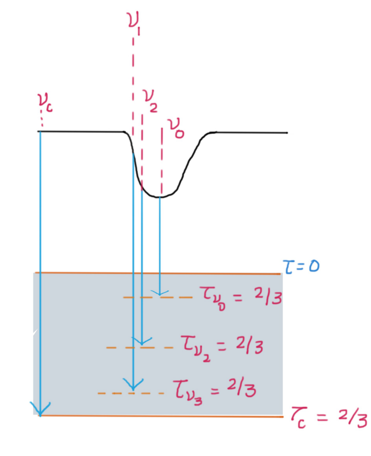

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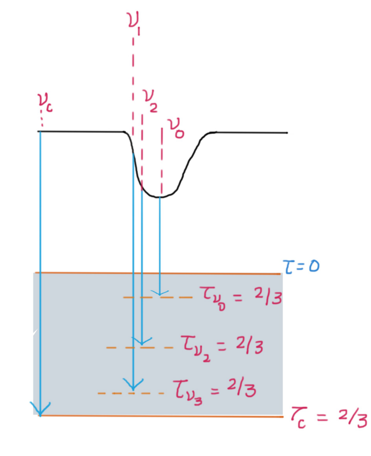

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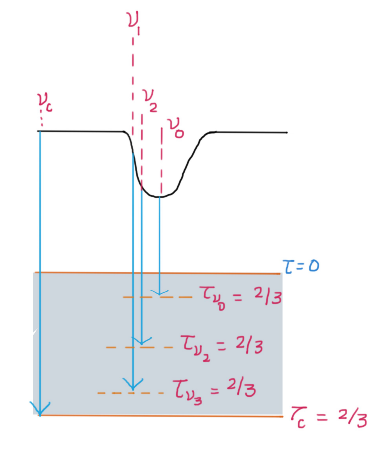

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Instructive results can be obtained for some semi-analytical models with simplifying assumptions.

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Photons continually removed from the column under consideration, and none survive for $\sigma_\ell^{\rm sca}/\sigma_c^{\rm abs} \rightarrow \infty$.

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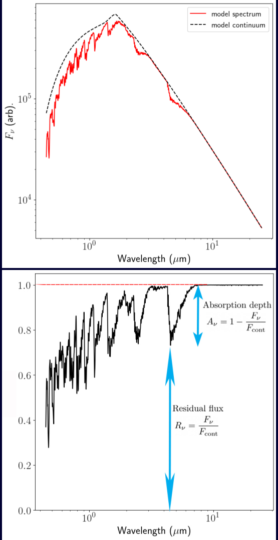
Strong abs. $\Rightarrow \sigma_\ell^{\mathrm{abs}} / \sigma_c^{\mathrm{abs}} \rightarrow \infty \Rightarrow R_\nu(0) \rightarrow \dfrac{1}{1 + b_\nu / \sqrt{3} a_\nu}$. Core residual flux of a strong abs. line stays finite.

If no temperature gradient, $b_\nu = 0$ and $R_\nu(0) = 1$ (line disappears).

Recall: equivalent width



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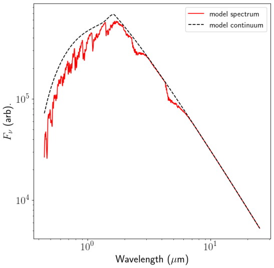

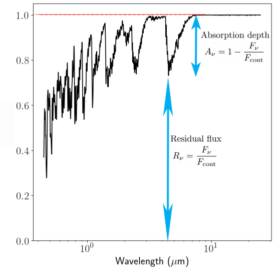

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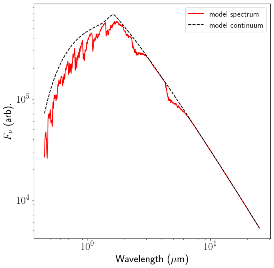

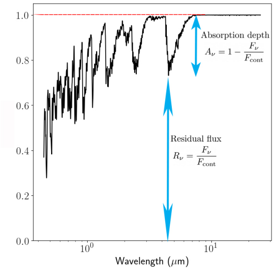

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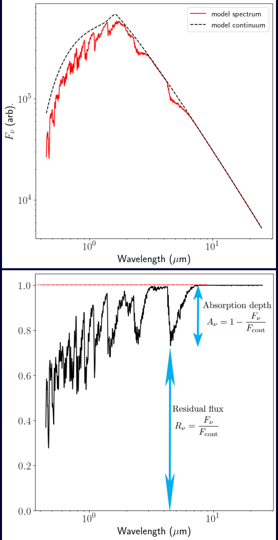

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How does EQW vary with abundance?

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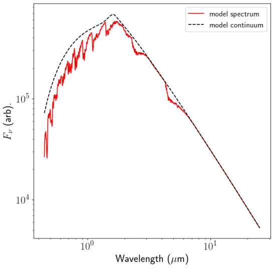

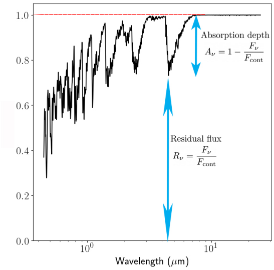

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Curve of Growth – can be determined theoretically or empirically.

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$$H(a,u) = \frac{a}{\pi}\int \frac{e^{-y^2}\,dy}{(u-y)^2 + a^2} \approx \overbrace{e^{-u^2}}^{\text{core}} + \overbrace{\frac{a}{\sqrt{\pi}u^2}}^{\text{wings}}; \quad u = \frac{\nu - \nu_0}{\Delta\nu_D},\ a = \frac{\Gamma}{4\pi\Delta\nu_D},\ \Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$$

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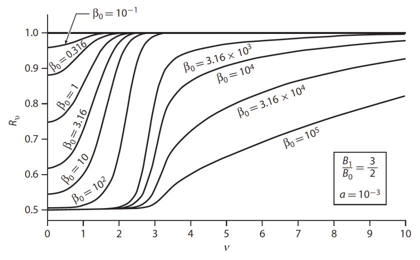
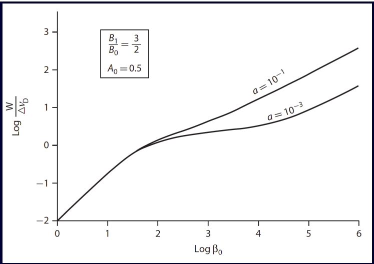
$$EQW = EQW_{\rm core} + 2\int_{u_2}^{\infty} \frac{\tau_0 a}{\sqrt{\pi}u^2}\, du = \frac{2\tau_0 a}{\sqrt{\pi}u_2} = \frac{2\tau_0 a}{\sqrt{\pi}}\sqrt{\frac{\sqrt{\pi}}{\tau_0 a}} \propto \sqrt{\tau_0 a} \propto \sqrt{N_1}.$$ Damping or square-root regime.

Theoretical Curve of Growth (Milne-Eddington Model)





from Hubeny & Mihalas Ch 17

Curve of growth depends on damping factor a.

Note that saturation sets in around $W \approx \Delta\nu_D$, and that wings start contributing when $\beta_\nu \sim a^{-1}$.

As $a \uparrow$, wings dominate faster.

Empirical Curve of Growth

$$\tau_\nu = N_1 \sigma_{12}; \quad \text{here, } \frac{N_1}{N} = \frac{g_1}{u(T)} \exp\left[-\frac{E_1}{kT}\right] \text{ and } \sigma_{12} = \frac{\sqrt{\pi} r_e c^2}{\nu_0} \sqrt{\frac{m}{2kT}} f_{12}$$

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X axis: τ_ν (theoretical) vs. $\frac{g_1 f_{12}}{\nu_0}$ (experimental).

Since $T_{\rm ex}$ is known, comparing the two abscissae gives us N.