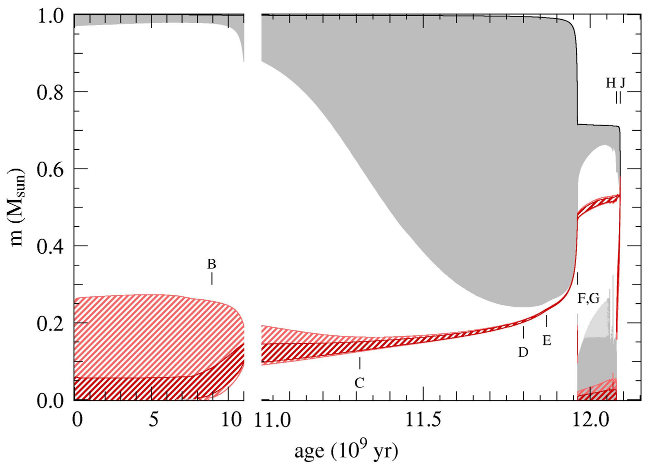
STELLAR STRUCTURE AND EVOLUTION



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Preface

These lecture notes are intended for an advanced astrophysics course on Stellar Structure and Evolution given at Utrecht University (NS-AP434M). Their goal is to provide an overview of the physics of stellar interiors and its application to the theory of stellar structure and evolution, at a level appropriate for a third-year Bachelor student or beginning Master student in astronomy. To a large extent these notes draw on the classical textbook by Kippenhahn & Weigert (1990; see below), but leaving out unnecessary detail while incorporating recent astrophysical insights and up-to-date results. At the same time I have aimed to concentrate on physical insight rather than rigorous derivations, and to present the material in a logical order, following in part the very lucid but somewhat more basic textbook by Prialnik (2000). Finally, I have borrowed some ideas from the textbooks by Hansen, Kawaler & Trimble (2004), Salaris & Cassissi (2005) and the recent book by Maeder (2009).

These lecture notes are evolving and I try to keep them up to date. If you find any errors or inconsistencies, I would be grateful if you could notify me by email (`O.R.Pols@uu.nl`).

Onno Pols
Utrecht, September 2011

Literature

- C.J. Hansen, S.D. Kawaler & V. Trimble, *Stellar Interiors*, 2004, Springer-Verlag, ISBN 0-387-20089-4 (Hansen)
- R. Kippenhahn & A. Weigert, *Stellar Structure and Evolution*, 1990, Springer-Verlag, ISBN 3-540-50211-4 (Kippenhahn; K&W)
- A. Maeder, *Physics, Formation and Evolution of Rotating Stars*, 2009, Springer-Verlag, ISBN 978-3-540-76948-4 (Maeder)
- D. Prialnik, *An Introduction to the Theory of Stellar Structure and Evolution*, 2nd edition, 2009, Cambridge University Press, ISBN 0-521-86604-9 (Prialnik)
- M. Salaris & S. Cassisi, *Evolution of Stars and Stellar Populations*, 2005, John Wiley & Sons, ISBN 0-470-09220-3 (Salaris)

Physical and astronomical constants

Table 1. Physical constants in cgs units (CODATA 2006).

gravitational constant	G	$6.674\,3 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
speed of light in vacuum	c	$2.997\,924\,58 \times 10^{10}$ cm s^{-1}
Planck constant	h	$6.626\,069 \times 10^{-27}$ erg s
radiation density constant	a	$7.565\,78 \times 10^{-15}$ erg cm^{-3} K^{-4}
Stefan-Boltzmann constant	$\sigma = \frac{1}{4}ac$	$5.670\,40 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}
Boltzmann constant	k	$1.380\,650 \times 10^{-16}$ erg K^{-1}
Avogadro's number	$N_A = 1/m_u$	$6.022\,142 \times 10^{23}$ g^{-1}
gas constant	$\mathcal{R} = kN_A$	$8.314\,47 \times 10^{7}$ erg g^{-1} K^{-1}
electron volt	eV	$1.602\,176\,5 \times 10^{-12}$ erg
electron charge	e	$4.803\,26 \times 10^{-10}$ esu
	e^2	$1.440\,00 \times 10^{-7}$ eV cm
electron mass	m_e	$9.109\,382 \times 10^{-28}$ g
atomic mass unit	m_u	$1.660\,538\,8 \times 10^{-24}$ g
proton mass	m_p	$1.672\,621\,6 \times 10^{-24}$ g
neutron mass	m_n	$1.674\,927\,2 \times 10^{-24}$ g
α-particle mass	m_α	$6.644\,656\,2 \times 10^{-24}$ g

Table 2. Astronomical constants, mostly from the Astronomical Almanac (2008).

Solar mass	$M_\odot$	$1.988\,4 \times 10^{33}$ g
	$GM_\odot$	$1.327\,124\,42 \times 10^{26}$ cm^3 s^{-2}
Solar radius	$R_\odot$	6.957×10^{10} cm
Solar luminosity	$L_\odot$	3.842×10^{33} erg s^{-1}
year	yr	$3.155\,76 \times 10^{7}$ s
astronomical unit	AU	$1.495\,978\,71 \times 10^{13}$ cm
parsec	pc	$3.085\,678 \times 10^{18}$ cm

Chapter 1

Introduction

This introductory chapter sets the stage for the course, and briefly repeats some concepts from earlier courses on stellar astrophysics (e.g. the Utrecht first-year course *Introduction to stellar structure and evolution* by F. Verbunt).

The *goal* of this course on stellar evolution can be formulated as follows:

> to understand the structure and evolution of stars, and their observational properties, using known laws of physics

This involves applying and combining 'familiar' physics from many different areas (e.g. thermodynamics, nuclear physics) under extreme circumstances (high temperature, high density), which is part of what makes studying stellar evolution so fascinating.

What exactly do we mean by a 'star'? A useful definition for the purpose of this course is as follows: a star is an object that (1) radiates energy from an internal source and (2) is bound by its own gravity. This definition excludes objects like planets and comets, because they do not comply with the first criterion. In the strictest sense it also excludes brown dwarfs, which are not hot enough for nuclear fusion, although we will briefly discuss these objects. (The second criterion excludes trivial objects that radiate, e.g. glowing coals).

An important implication of this definition is that stars must *evolve* (why?). A star is born out of an interstellar (molecular) gas cloud, lives for a certain amount of time on its internal energy supply, and eventually dies when this supply is exhausted. As we shall see, a second implication of the definition is that stars can have only a limited range of masses, between ∼0.1 and ∼100 times the mass of the Sun. The *life and death* of stars forms the subject matter of this course. We will only briefly touch on the topic of *star formation*, a complex and much less understood process in which the problems to be solved are mostly very different than in the study of stellar evolution.

1.1 Observational constraints

Fundamental properties of a star include the *mass* M (usually expressed in units of the solar mass, $M_\odot = 1.99 \times 10^{33}$ g), the *radius* R (often expressed in $R_\odot = 6.96 \times 10^{10}$ cm) and the *luminosity* L, the rate at which the star radiates energy into space (often expressed in $L_\odot = 3.84 \times 10^{33}$ erg/s). The *effective temperature* $T_{\rm eff}$ is defined as the temperature of a black body with the same energy flux at the surface of the star, and is a good measure for the temperature of the photosphere. From the definition of effective temperature it follows that

$$L = 4\pi R^2 \sigma T_{\rm eff}^4. \tag{1.1}$$

In addition, we would like to know the *chemical composition* of a star. Stellar compositions are usually expressed as mass fractions X_i, where i denotes a certain element. This is often simplified to specifying the mass fractions X (of hydrogen), Y (of helium) and Z (of all heavier elements or 'metals'), which add up to unity. Another fundamental property is the *rotation rate* of a star, expressed either in terms of the rotation period $P_{\rm rot}$ or the equatorial rotation velocity $\upsilon_{\rm eq}$.

Astronomical observations can yield information about these fundamental stellar quantities:

- *Photometric measurements* yield the apparent brightness of a star, i.e. the energy flux received on Earth, in different wavelength bands. These are usually expressed as magnitudes, e.g. B, V, I, etc. Flux ratios or colour indices ($B-V$, $V-I$, etc.) give a measure of the effective temperature, using theoretical stellar atmosphere models and/or empirical relations. Applying a bolometric correction (which also depends on $T_{\rm eff}$) yields the apparent bolometric flux, $f_{\rm bol}$ (in erg s^{-1} cm^{-2}).

- In some cases the *distance d* to a star can be measured, e.g. from the parallax. The Hipparcos satellite has measured parallaxes with 1 milliarcsec accuracy of more than 10^5 stars. The luminosity then follows from $L = 4\pi d^2 f_{\rm bol}$, and the radius from eq. (1.1) if we have a measure of $T_{\rm eff}$.

- An independent measure of the effective temperature can be obtained from *interferometry*. This technique yields the angular diameter of a star if it is sufficiently extended on the sky, i.e. the ratio $\theta = R/d$. Together with a measurement of $f_{\rm bol}$ this can be seen from eq. (1.1) to yield $\sigma T_{\rm eff}^4 = f_{\rm bol}/\theta^2$. This technique is applied to red giants and supergiants. If the distance is also known, a direct measurement of the radius is possible.

- *Spectroscopy* at sufficiently high resolution yields detailed information about the physical conditions in the atmosphere. With detailed spectral-line analysis using stellar atmosphere models one can determine the photospheric properties of a star: the effective temperature and surface gravity ($g = GM/R^2$, usually expressed as $\log g$), surface abundances of various elements (usually in terms of number density relative to hydrogen) and a measure of the rotation velocity ($\upsilon_{\rm eq} \sin i$, where i is the unknown inclination angle of the equatorial plane). In addition, for some stars the properties of the *stellar wind* can be determined (wind velocities, mass loss rates). All this is treated in more detail in the Master course on *Stellar Atmospheres*.

- The most important fundamental property, the mass, cannot be measured directly for a single star. To measure stellar masses one needs *binary stars* showing radial velocity variations (spectroscopic binaries). Radial velocities alone can only yield masses up to a factor $\sin i$, where i is the inclination angle of the binary orbit. To determine absolute mass values one needs information on i, either from a visual orbit (visual binaries) or from the presence of eclipses (eclipsing binaries). In particular for so called double-lined eclipsing binaries, in which the spectral lines of both stars vary, it is possible to accurately measure both the masses and radii (with 1–2 % accuracy in some cases) by fitting the radial-velocity curves and the eclipse lightcurve. Together with a photometric or, better, spectroscopic determination of $T_{\rm eff}$ also the luminosity of such binaries can be measured with high accuracy, independent of the distance. For more details see the Master course on *Binary Stars*.

All observed properties mentioned above are surface properties. Therefore we need a *theory of stellar structure* to derive the internal properties of a star. However, some direct windows on the interior of a star exist:

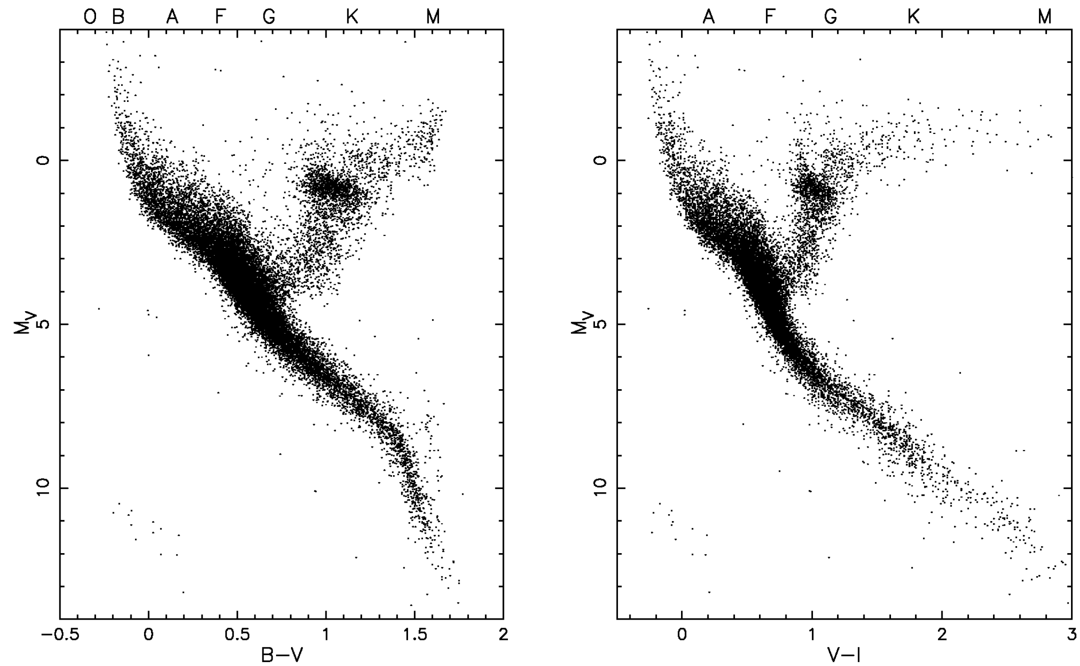

Figure 1.1. H-R diagram of solar neighbourhood. Source: Hipparcos, stars with d measured to $< 10\,\%$ accuracy.

- *neutrinos*, which escape from the interior without interaction. So far, the Sun is the only (non-exploding) star from which neutrinos have been detected.
- *oscillations*, i.e. stellar seismology. Many stars oscillate, and their frequency spectrum contains information about the speed of sound waves inside the star, and therefore about the interior density and temperature profiles. This technique has provided accurate constraints on detailed structure models for the Sun, and is now also being applied to other stars.

The timespan of any observations is much smaller than a stellar lifetime: observations are like snapshots in the life of a star. The observed properties of an individual star contain no (direct) information about its evolution. The diversity of stellar properties (radii, luminosities, surface abundances) does, however, depend on how stars evolve, as well as on intrinsic properties (mass, initial composition). Properties that are common to a large number of stars must correspond to long-lived evolution phases, and vice versa. By studying samples of stars statistically we can infer the (relative) lifetimes of certain phases, which provides another important constraint on the theory of stellar evolution.

Furthermore, observations of samples of stars reveal certain correlations between stellar properties that the theory of stellar evolution must explain. Most important are relations between luminosity and effective temperature, as revealed by the *Hertzsprung-Russell diagram*, and relations between mass, luminosity and radius.

1.1.1 The Hertzsprung-Russell diagram

The Hertzsprung-Russell diagram (HRD) is an important tool to test the theory of stellar evolution. Fig. 1.1 shows the colour-magnitude diagram (CMD) of stars in the vicinity of the Sun, for which the Hipparcos satellite has measured accurate distances. This is an example of a *volume-limited* sample

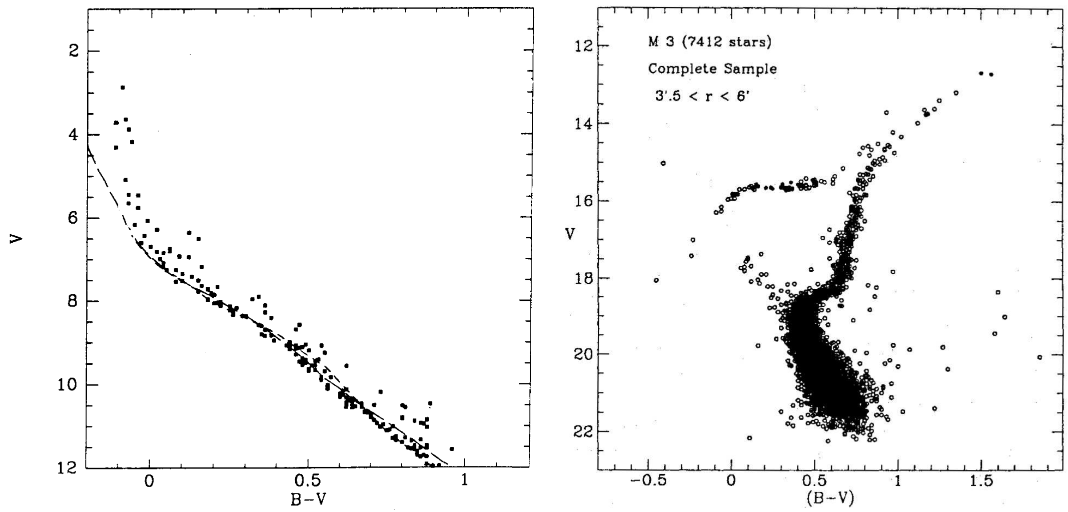

Figure 1.2. Colour-magnitude diagrams of a young open cluster, M45 (the Pleiades, left panel), and a globular cluster, M3 (right panel).

of stars. In this observers' HRD, the absolute visual magnitude M_V is used as a measure of the luminosity and a colour index ($B - V$ or $V - I$) as a measure for the effective temperature. It is left as an exercise to identify various types of stars and evolution phases in this HRD, such as the main sequence, red giants, the horizontal branch, white dwarfs, etc.

Star clusters provide an even cleaner test of stellar evolution. The stars in a cluster were formed within a short period of time (a few Myr) out of the same molecular cloud and therefore share the same age and (initial) chemical composition.[1] Therefore, to first-order approximation only the mass varies from star to star. A few examples of cluster CMDs are given in Fig. 1.2, for a young open cluster (the Pleiades) and an old globular cluster (M3). As the cluster age increases, the most luminous main-sequence stars disappear and a prominent red giant branch and horizontal branch appear. To explain the morphology of cluster HRDs at different ages is one of the goals of studying stellar evolution.

1.1.2 The mass-luminosity and mass-radius relations

For stars with measured masses, radii and luminosities (i.e. binary stars) we can plot these quantities against each other. This is done in Fig. 1.3 for the components of double-lined eclipsing binaries for which M, R and L are all measured with $\lesssim 2\,\%$ accuracy. These quantities are clearly correlated, and especially the relation between mass and luminosity is very tight. Most of the stars in Fig. 1.3 are long-lived main-sequence stars; the spread in radii for masses between 1 and $2\,M_\odot$ results from the fact that several more evolved stars in this mass range also satisfy the $2\,\%$ accuracy criterion. The observed relations can be approximated reasonably well by power laws:

$$L \propto M^{3.8} \qquad \text{and} \qquad R \propto M^{0.7}. \tag{1.2}$$

Again, the theory of stellar evolution must explain the existence and slopes of these relations.

[1]The stars in a cluster thus consitute a so-called *simple stellar population*. Recently, this simple picture has changed somewhat after the discovery of multiple populations in many star clusters.

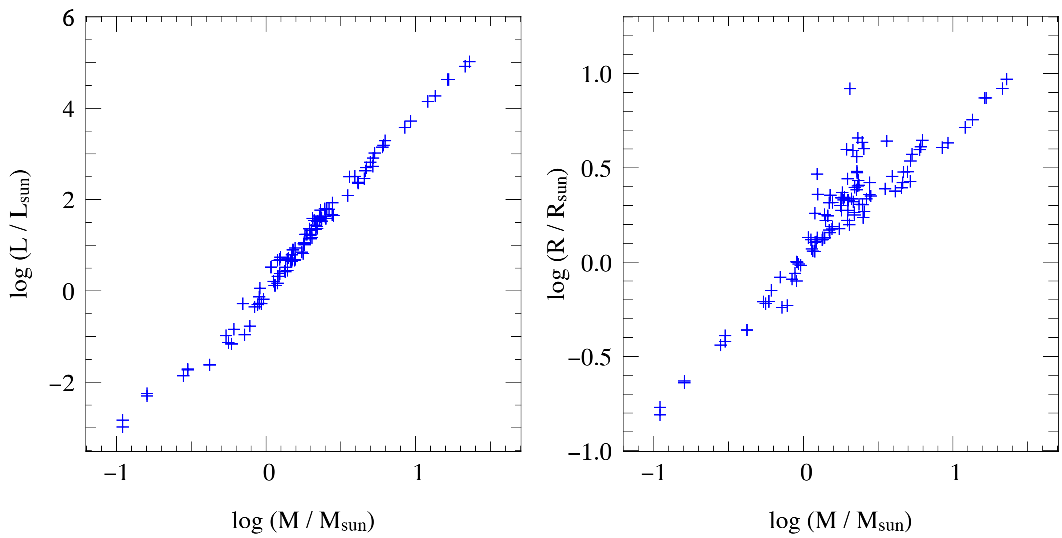

Figure 1.3. Mass-luminosity (left) and mass-radius (right) relations for components of double-lined eclipsing binaries with accurately measured M, R and L.

1.2 Stellar populations

Stars in the Galaxy are divided into different populations:

- Population I: stars in the galactic disk, in spiral arms and in (relatively young) open clusters. These stars have ages $\lesssim 10^9$ yr and are relatively metal-rich ($Z \sim 0.5 - 1\, Z_\odot$)

- Population II: stars in the galactic halo and in globular clusters, with ages $\sim 10^{10}$ yr. These stars are observed to be metal-poor ($Z \sim 0.01 - 0.1\, Z_\odot$).

An intermediate population (with intermediate ages and metallicities) is also seen in the disk of the Galaxy. Together they provide evidence for the *chemical evolution* of the Galaxy: the abundance of heavy elements (Z) apparently increases with time. This is the result of chemical enrichment by subsequent stellar generations.

The study of chemical evolution has led to the hypothesis of a 'Population III' consisting of the first generation of stars formed after the Big Bang, containing only hydrogen and helium and no heavier elements ('metal-free', $Z = 0$). No metal-free stars have ever been observed, probably due to the fact that they were massive and had short lifetimes and quickly enriched the Universe with metals. However, a quest for finding their remnants has turned up many very metal-poor stars in the halo, with the current record-holder having an iron abundance $X_{\rm Fe} = 4 \times 10^{-6} X_{\rm Fe,\odot}$.

1.3 Basic assumptions

We wish to build a theory of stellar evolution to explain the observational constraints highlighted above. In order to do so we must make some basic assumptions:

- stars are considered to be *isolated* in space, so that their structure and evolution depend only on *intrinsic* properties (mass and composition). For most single stars in the Galaxy this condition is satisfied to a high degree (compare for instance the radius of the Sun with the distance to its nearest neighbour Proxima Centauri, see exercise 1.2). However, for stars in dense clusters, or in binary systems, the evolution can be influenced by interaction with neighbouring stars. In this course we will mostly ignore these complicating effects (many of which are treated in the Master course on *Binary Stars*).

- stars are formed with a *homogeneous composition*, a reasonable assumption since the molecular clouds out of which they form are well-mixed. We will often assume a so-called 'quasi-solar' composition ($X = 0.70$, $Y = 0.28$ and $Z = 0.02$), even though recent determinations of solar abundances have revised the solar metallicity down to $Z = 0.014$. In practice there is relatively little variation in composition from star to star, so that the initial mass is the most important parameter that determines the evolution of a star. The composition, in particular the metallicity Z, is of secondary influence but can have important effects especially in very metal-poor stars (see § 1.2).

- *spherical symmetry*, which is promoted by self-gravity and is a good approximation for most stars. Deviations from spherical symmetry can arise if non-central forces become important relative to gravity, in particular rotation and magnetic fields. Although many stars are observed to have magnetic fields, the field strength (even in highly magnetized neutron stars) is always negligible compared to gravity. Rotation can be more important, and the *rotation rate* can be considered an additional parameter (besides mass and composition) determining the structure and evolution of a star. For the majority of stars (e.g. the Sun) the forces involved are small compared to gravity. However, some rapidly rotating stars are seen (by means of interferometry) to be substantially flattened.

1.4 Aims and overview of the course

In the remainder of this course we will:

- understand the global properties of stars: energetics and timescales
- study the micro-physics relevant for stars: the equation of state, nuclear reactions, energy transport and opacity
- derive the equations necessary to model the internal structure of stars
- examine (quantitatively) the properties of simplified stellar models
- survey (mostly qualitatively) how stars of different masses evolve, and the endpoints of stellar evolution (white dwarfs, neutron stars)
- discuss a few ongoing research areas in stellar evolution

Suggestions for further reading

The contents of this introductory chapter are also largely covered by Chapter 1 of Prialnik, which provides nice reading.

Exercises

1.1 Evolutionary stages

In this course we use many concepts introduced in the introductory astronomy classes. In this exercise we recapitulate the names of evolutionary phases. During the lectures you are assumed to be familiar with these terms, in the sense that you are able to explain them in general terms.

We encourage you to use Carroll & Ostlie, *Introduction to Modern Astrophysics*, or the book of the first year course (Verbunt, *Het leven van sterren*) to make a list of the concepts printed in *italic* with a brief explanation in your own words.

(a) Figure 1.1 shows the location of stars in the solar neighborhood in the Hertzsprung-Russel diagram. Indicate in Figure 1.1 where you would find:

main-sequence stars,	*neutron stars,*
the Sun,	*black holes,*
red giants,	*binary stars,*
horizontal branch stars,	*planets,*
asymptotic giant branch (AGB) stars,	*pre-main sequence stars,*
centrals star of planetary nebulae,	*hydrogen burning stars,*
white dwarfs,	*helium burning stars.*

(b) Through which stages listed above will the Sun evolve? Put them in chronological order. Through which stages will a massive star evolve?

(c) Describe the following concepts briefly in your own words. You will need the concepts indicated with * in the coming lectures.

ideal gas,*	*Jeans mass,*
black body,	*Schwarzschild criterion,*
virial theorem,*	*energy transport by radiation,*
first law of thermodynamics,*	*energy transport by convection,*
equation of state,	*pp-chain,*
binary stars,	*CNO cycle,*
star cluster,	*nuclear timescale*,*
interstellar medium,	*thermal or Kelvin-Helmholtz timescale*,*
giant molecular clouds,	*dynamical timescale**

1.2 Basic assumptions

Let us examine the three basic assumptions made in the theory of stellar evolution:

(a) *Stars are assumed to be isolated in space.* The star closest to the sun, Proxima Centauri, is 4.3 light-years away. How many solar radii is that? By what factors are the gravitational field and the radiation flux diminished? Many stars are formed in clusters and binaries. How could that influence the life of a star?

(b) *Stars are assumed to form with a uniform composition.* What elements is the Sun made of? Just after the Big Bang the Universe consisted almost purely of hydrogen and helium. Where do all the heavier elements come from?

(c) *Stars are assumed to be spherically symmetric.* Why are stars spherically symmetric to a good approximation? How would rotation affect the structure and evolution of a star? The Sun rotates around its axis every 27 days. Calculate the ratio of is the centrifugal acceleration a over the gravitational acceleration g for a mass element on the surface of the Sun. Does rotation influence the structure of the Sun?

1.3 Mass-luminosity and mass-radius relation

(a) The masses of stars are approximately in the range $0.08\,M_\odot \lesssim M \lesssim 100\,M_\odot$. Why is there an upper limit? Why is there a lower limit?

(b) Can you think of methods to measure (1) the mass, (2) the radius, and (3) the luminosity of a star? Can your methods be applied for any star or do they require special conditions. Discuss your methods with your fellow students.

(c) Figure 1.3 shows the luminosity versus the mass (left) and the radius versus the mass (right) for observed main sequence stars. We can approximate a mass-luminosity and mass-radius relation by fitting functions of the form

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^x, \qquad \frac{R}{L_\odot} = \left(\frac{M}{M_\odot}\right)^y \tag{1.3}$$

Estimate x and y from Figure 1.3.

(d) Which stars live longer, high mass stars (which have more fuel) or low mass stars? Derive an expression for the lifetime of a star as a function of its mass. (!)

[Hints: Stars spend almost all their life on the main sequence burning hydrogen until they run out of fuel. First try to estimate the life time as function of the mass (amount of fuel) and the luminosity (rate at which the fuel is burned).]

1.4 The ages of star clusters

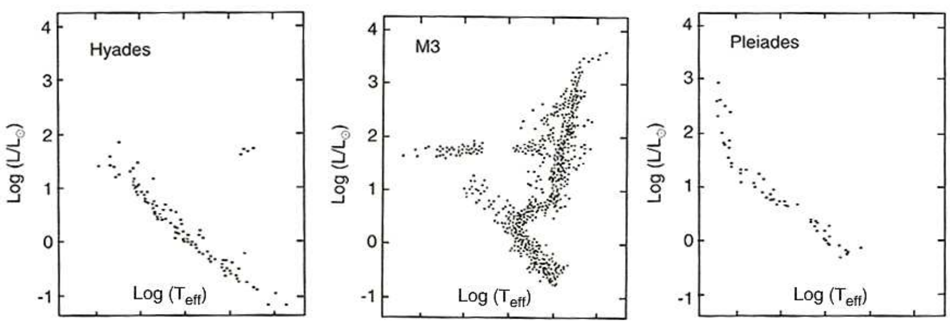

Figure 1.4. H-R diagrams of three star clusters (from Prialnik).

The stars in a star cluster are formed more or less simultaneously by fragmentation of a large molecular gas cloud.

(a) In Fig. 1.4 the H-R diagrams are plotted of the stars in three different clusters. Which cluster is the youngest?

(b) Think of a method to estimate the age of the clusters, discuss with your fellow students. Estimate the ages and compare with the results of your fellow students.

(c) (*) Can you give an error range on your age estimates?

Chapter 2

Mechanical and thermal equilibrium

In this chapter we apply the physical principles of mass conservation and momentum conservation to derive two of the fundamental stellar structure equations. We shall see that stars are generally in a state of almost complete *mechanical equilibrium*, which allows us to derive and apply the important *virial theorem*. We consider the basic stellar timescales and see that most (but not all) stars are also in a state of energy balance called *thermal equilibrium*.

2.1 Coordinate systems and the mass distribution

The assumption of spherical symmetry implies that all interior physical quantities (such as density ρ, pressure P, temperature T, etc) depend only on one radial coordinate. The obvious coordinate to use in a Eulerian coordinate system is the radius of a spherical shell, r ($\in 0 \ldots R$). In an evolving star, all quantities also depend on time t. When constructing the differential equations for stellar structure one should thus generally consider partial derivatives of physical quantities with respect to radius and time, $\partial/\partial r$ and $\partial/\partial t$, taken at constant t and r, respectively.

The principle of mass conservation applied to the mass dm of a spherical shell of thickness dr at radius r (see Fig. 2.1) gives

$$\mathrm{d}m(r,t) = 4\pi r^2 \rho\,\mathrm{d}r - 4\pi r^2 \rho\,\upsilon\,\mathrm{d}t, \tag{2.1}$$

where υ is the radial velocity of the mass shell. Therefore one has

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \qquad \text{and} \qquad \frac{\partial m}{\partial t} = -4\pi r^2 \rho\,\upsilon. \tag{2.2}$$

The first of these partial differential equations relates the radial mass distribution in the star to the local density: it constitutes the first fundamental equation of stellar structure. Note that $\rho = \rho(r,t)$ is not known a priori, and must follow from other conditions and equations. The second equation of (2.2) represents the change of mass inside a sphere of radius r due to the motion of matter through its surface; at the stellar surface this gives the mass-loss rate (if there is a stellar wind with $\upsilon > 0$) or mass-accretion rate (if there is inflow with $\upsilon < 0$). In a static situation, where the velocity is zero, the first equation of (2.2) becomes an ordinary differential equation,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho. \tag{2.3}$$

This is almost always a good approximation for stellar interiors, as we shall see. Integration yields the mass $m(r)$ inside a spherical shell of radius r:

$$m(r) = \int_0^r 4\pi r'^2 \rho\,\mathrm{d}r'.$$

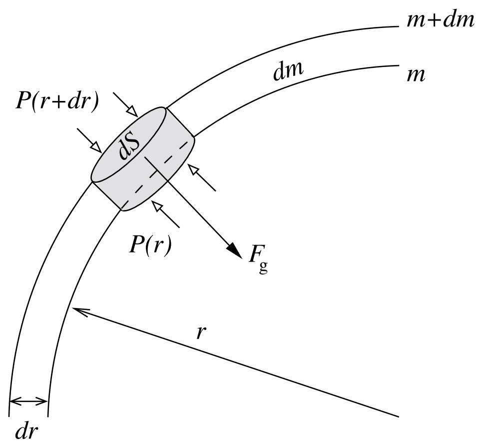

Figure 2.1. Mass shell inside a spherically symmetric star, at radius r and with thickness dr. The mass of the shell is $\mathrm{d}m = 4\pi r^2\rho\,\mathrm{d}r$. The pressure and the gravitational force acting on a cylindrical mass element are also indicated.

Since $m(r)$ increases monotonically outward, we can also use $m(r)$ as our radial coordinate, instead of r. This *mass coordinate*, often denoted as m_r or simply m, is a Lagrangian coordinate that moves with the mass shells:

$$m := m_r = \int_0^r 4\pi r'^2 \rho\,\mathrm{d}r' \qquad (m \in 0 \ldots M). \tag{2.4}$$

It is often more convenient to use a Lagrangian coordinate instead of a Eulerian coordinate. The mass coordinate is defined on a fixed interval, $m \in 0 \ldots M$, as long as the star does not lose mass. On the other hand r depends on the time-varying stellar radius R. Furthermore the mass coordinate follows the mass elements in the star, which simplifies many of the time derivatives that appear in the stellar evolution equations (e.g. equations for the composition). We can thus write all quantities as functions of m, i.e. $r = r(m)$, $\rho = \rho(m)$, $P = P(m)$, etc.

Using the coordinate transformation $r \to m$, i.e.

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial m}, \tag{2.5}$$

the first equation of stellar structure becomes in terms of the coordinate m:

$$\boxed{\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}} \tag{2.6}$$

which again becomes an ordinary differential equation in a static situation.

2.1.1 The gravitational field

Recall that a star is a self-gravitating body of gas, which implies that gravity is the driving force behind stellar evolution. In the general, non-spherical case, the gravitational acceleration $\boldsymbol{g}$ can be written as the gradient of the gravitational potential, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, where Φ is the solution of the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho.$$

Inside a spherically symmetric body, this reduces to $g := |\boldsymbol{g}| = \mathrm{d}\Phi/\mathrm{d}r$. The gravitational acceleration at radius r and equivalent mass coordinate m is then given by

$$g = \frac{Gm}{r^2}. \tag{2.7}$$

Spherical shells outside r apply no net force, so that g only depends on the mass distribution inside the shell at radius r. Note that g is the magnitude of the vector $\boldsymbol{g}$ which points inward (toward smaller r or m).

2.2 The equation of motion and hydrostatic equilibrium

We next consider conservation of momentum inside a star, i.e. Newton's second law of mechanics. The net acceleration on a gas element is determined by the sum of all forces acting on it. In addition to the gravitational force considered above, forces result from the pressure exerted by the gas surrounding the element. Due to spherical symmetry, the pressure forces acting horizontally (perpendicular to the radial direction) balance each other and only the pressure forces acting along the radial direction need to be considered. By assumption we ignore other forces that might act inside a star (Sect. 1.3).

Hence the net acceleration $\ddot{r} = \partial^2 r/\partial t^2$ of a (cylindrical) gas element with mass

$$\mathrm{d}m = \rho\,\mathrm{d}r\,\mathrm{d}S \tag{2.8}$$

(where dr is its radial extent and dS is its horizontal surface area, see Fig. 2.1) is given by

$$\ddot{r}\,\mathrm{d}m = -g\,\mathrm{d}m + P(r)\,\mathrm{d}S - P(r+\mathrm{d}r)\,\mathrm{d}S. \tag{2.9}$$

We can write $P(r+\mathrm{d}r) = P(r) + (\partial P/\partial r)\cdot\mathrm{d}r$, hence after substituting eqs. (2.7) and (2.8) we obtain the *equation of motion* for a gas element inside the star:

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{2.10}$$

This is a simplified from of the Navier-Stokes equation of hydrodynamics, applied to spherical symmetry (see Maeder). Writing the pressure gradient $\partial P/\partial r$ in terms of the mass coordinate m by substituting eq. (2.6), the equation of motion is

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - 4\pi r^2\frac{\partial P}{\partial m}. \tag{2.11}$$

Hydrostatic equilibrium The great majority of stars are obviously in such long-lived phases of evolution that no change can be observed over human lifetimes. This means there is no noticeable acceleration, and all forces acting on a gas element inside the star almost exactly balance each other. Thus most stars are in a state of mechanical equilibrium which is more commonly called *hydrostatic equilibrium* (HE).

The state of hydrostatic equilibrium, setting $\ddot{r} = 0$ in eq. (2.10), yields the second differential equation of stellar structure:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm}{r^2}\rho, \tag{2.12}$$

or with eq. (2.6)

$$\boxed{\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}} \tag{2.13}$$

A direct consequence is that inside a star in hydrostatic equilibrium, the pressure always decreases outwards.

Eqs. (2.6) and (2.13) together determine the *mechanical structure* of a star in HE. These are two equations for three unknown functions of m (r, P and ρ), so they cannot be solved without a third condition. This condition is usually a relation between P and ρ called the *equation of state* (see Chapter 3). In general the equation of state depends on the temperature T as well, so that the mechanical structure depends also on the temperature distribution inside the star, i.e. on its thermal structure. In special cases the equation of state is independent of T, and can be written as $P = P(\rho)$. In such cases (known as barotropes or polytropes) the mechanical structure of a star becomes independent of its thermal structure. This is the case for white dwarfs, as we shall see later.

Estimates of the central pressure A rough order-of-magnitude estimate of the central pressure can be obtained from eq. (2.13) by setting

$$\frac{\mathrm{d}P}{\mathrm{d}m} \sim \frac{P_{\mathrm{surf}} - P_c}{M} \approx -\frac{P_c}{M}, \quad m \sim \tfrac{1}{2}M, \quad r \sim \tfrac{1}{2}R$$

which yields

$$P_c \sim \frac{2}{\pi}\frac{GM^2}{R^4} \tag{2.14}$$

For the Sun we obtain from this estimate $P_c \sim 7 \times 10^{15}\,\mathrm{dyn/cm^2} = 7 \times 10^9$ atm.

A lower limit on the central pressure may be derived by writing eq. (2.13) as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm}{4\pi r^4}\frac{\mathrm{d}m}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Gm^2}{8\pi r^4}\right) - \frac{Gm^2}{2\pi r^5},$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(P + \frac{Gm^2}{8\pi r^4}\right) = -\frac{Gm^2}{2\pi r^5} < 0. \tag{2.15}$$

The quantity $\Psi(r) = P + Gm^2/(8\pi r^4)$ is therefore a decreasing function of r. At the centre, the second term vanishes because $m \propto r^3$ for small r, and hence $\Psi(0) = P_c$. At the surface, the pressure is essentially zero. From the fact that Ψ must decrease with r it thus follows that

$$P_c > \frac{1}{8\pi}\frac{GM^2}{R^4}. \tag{2.16}$$

In contrast to eq. (2.14), this is a strict mathematical result, valid for any star in hydrostatic equilibrium regardless of its other properties (in particular, regardless of its density distribution). For the Sun we obtain $P_c > 4.4 \times 10^{14}\,\mathrm{dyn/cm^2}$. Both estimates indicate that an extremely high central pressure is required to keep the Sun in hydrostatic equilibrium. Realistic solar models show the central density to be $2.4 \times 10^{17}\,\mathrm{dyn/cm^2}$.

2.2.1 The dynamical timescale

We can ask what happens if the state of hydrostatic equilibrium is violated: how fast do changes to the structure of a star occur? The answer is provided by the equation of motion, eq. (2.10). For example, suppose that the pressure gradient that supports the star against gravity suddenly drops. All mass shells are then accelerated inwards by gravity: the star starts to collapse in "free fall". We can approximate the resulting (inward) acceleration by

$$|\ddot{r}| \approx \frac{R}{{\tau_{\mathrm{ff}}}^2} \quad \Rightarrow \quad \tau_{\mathrm{ff}} \approx \sqrt{\frac{R}{|\ddot{r}|}}$$

where $\tau_{\rm ff}$ is the free-fall timescale that we want to determine. Since $-\ddot{r} = g \approx GM/R^2$ for the entire star, we obtain

$$\tau_{\rm ff} \approx \sqrt{\frac{R}{g}} \approx \sqrt{\frac{R^3}{GM}}. \tag{2.17}$$

Of course each mass shell is accelerated at a different rate, so this estimate should be seen as an average value for the star to collapse over a distance R. This provides one possible estimate for the *dynamical timescale* of the star. Another estimate can be obtained in a similar way by assuming that gravity suddenly disappears: this gives the timescale for the outward pressure gradient to explode the star, which is similar to the time it takes for a sound wave to travel from the centre to the surface of the star. If the star is close to HE, all these timescales have about the same value given by eq. (2.17). Since the average density $\bar{\rho} = 3M/(4\pi R^3)$, we can also write this (hydro)dynamical timescale as

$$\tau_{\rm dyn} \approx \sqrt{\frac{R^3}{GM}} \approx \tfrac{1}{2}\,(G\bar{\rho})^{-1/2}. \tag{2.18}$$

For the Sun we obtain a very small value of $\tau_{\rm dyn} \approx 1600$ sec or about half an hour (0.02 days). This is very much smaller than the age of the Sun, which is 4.6 Gyr or $\sim 1.5 \times 10^{17}$ sec, by 14 orders of magnitude. This result has several important consequences for the Sun and other stars:

- Any significant departure from hydrostatic equilibrium should very quickly lead to observable phenomena: either contraction or expansion on the dynamical timescale. If the star cannot recover from this disequilibrium by restoring HE, it should lead to a collapse or an explosion.
- Normally hydrostatic equilibrium can be restored after a disturbance (we will consider this *dynamical stability* of stars later). However a perturbation of HE may lead to small-scale oscillations on the dynamical timescale. These are indeed observed in the Sun and many other stars, with a period of minutes in the case of the Sun. Eq. (2.18) tells us that the pulsation period is a (rough) measure of the average density of the star.
- Apart from possible oscillations, stars are extremely close to hydrostatic equilibrium, since any disturbance is immediately quenched. We can therefore be confident that eq. (2.13) holds throughout most of their lifetimes. Stars do evolve and are therefore not completely static, but changes occur very slowly compared to their dynamical timescale. Stars can be said to evolve *quasi-statically*, i.e. through a series of near-perfect HE states.

2.3 The virial theorem

An important consequence of hydrostatic equilibrium is the *virial theorem*, which is of vital importance for the understanding of stars. It connects two important energy reservoirs of a star and allows predictions and interpretations of important phases in the evolution of stars.

To derive the virial theorem we start with the equation for hydrostatic equilibrium eq. (2.13). We multiply both sides by the enclosed volume $V = \frac{4}{3}\pi r^3$ and integrate over m:

$$\int_0^M \tfrac{4}{3}\pi r^3 \frac{{\rm d}P}{{\rm d}m}\,{\rm d}m = -\tfrac{1}{3}\int_0^M \frac{Gm}{r}\,{\rm d}m \tag{2.19}$$

The integral on the right-hand side has a straightforward physical interpretation: it is the *gravitational potential energy* of the star. To see this, consider the work done by the gravitational force $\boldsymbol{F}$ to bring a mass element δm from infinity to radius r:

$$\delta W = \int_\infty^r \boldsymbol{F}\cdot{\bf d}\boldsymbol{r} = \int_\infty^r \frac{Gm\,\delta m}{r^2}\,{\rm d}r = -\frac{GM}{r}\delta m.$$

The gravitational potential energy of the star is the work performed by the gravitational force to bring *all* mass elements from infinity to their current radius, i.e.

$$E_{\rm gr} = -\int_0^M \frac{Gm}{r}\,{\rm d}m \tag{2.20}$$

The left-hand side of eq. (2.19) can be integrated by parts:

$$\int_{P_c}^{P_s} V\,{\rm d}P = [V\cdot P]_{\rm c}^{\rm s} - \int_0^{V_s} P\,{\rm d}V \tag{2.21}$$

where c and s denote central and surface values. Combining the above expressions in eq. (2.19) we obtain

$$\tfrac{4}{3}\pi R^3\,P(R) - \int_0^{V_s} P\,{\rm d}V = \tfrac{1}{3}E_{\rm gr}, \tag{2.22}$$

with $P(R)$ the pressure at the surface of the volume. This expression is useful when the pressure from the surrounding layers is substantial, e.g. when we consider only the core of a star. If we consider the star as a whole, however, the first term vanishes because the pressure at the stellar surface is negligible. In that case

$$-3\int_0^{V_s} P\,{\rm d}V = E_{\rm gr}, \tag{2.23}$$

or, since ${\rm d}V = {\rm d}m/\rho$,

$$-3\int_0^M \frac{P}{\rho}\,{\rm d}m = E_{\rm gr}. \tag{2.24}$$

This is the general form of the virial theorem, which will prove valuable later. It tells us that that the average pressure needed to support a star in HE is equal to $-\frac{1}{3}E_{\rm gr}/V$. In particular it tells us that a star that contracts quasi-statically (that is, slowly enough to remain in HE) must increase its internal pressure, since $|E_{\rm gr}|$ increases while its volume decreases.

The virial theorem for an ideal gas The pressure of a gas is related to its internal energy. We will show this in Ch. 3, but for the particular case of an ideal monatomic gas it is easy to see. The pressure of an ideal gas is given by

$$P = nkT = \frac{\rho}{\mu m_{\rm u}}kT, \tag{2.25}$$

where $n = N/V$ is the number of particles per unit volume, and μ is mass of a gas particle in atomic mass units. The kinetic energy per particle is $\epsilon_k = \frac{3}{2}kT$, and the internal energy of an ideal monatomic gas is equal to the kinetic energy of its particles. The internal energy per unit mass is then

$$u = \frac{3}{2}\frac{kT}{\mu m_{\rm u}} = \frac{3}{2}\frac{P}{\rho}. \tag{2.26}$$

We can now interpret the left-hand side of the virial theorem (eq. 2.24) as $\int(P/\rho)\,{\rm d}m = \frac{2}{3}\int u\,{\rm d}m = \frac{2}{3}E_{\rm int}$, where $E_{\rm int}$ is the total internal energy of the star. The virial theorem for an ideal gas is therefore

$$\boxed{E_{\rm int} = -\tfrac{1}{2}E_{\rm gr}} \tag{2.27}$$

This important relation establishes a link between the gravitational potential energy and the internal energy of a star in hydrostatic equilibrium that consists of an ideal gas. (We shall see later that the ideal gas law indeed holds for most stars, at least on the main sequence.) The virial theorem tells us that a more tightly bound star must have a higher internal energy, i.e. it must be *hotter*. In other words, a star that contracts quasi-statically must get hotter in the process. The full implications of this result will become clear when we consider the total energy of a star in a short while.

Estimate of the central temperature Using the virial theorem we can obtain an estimate of the average temperature inside a star composed of ideal gas. The gravitational energy of the star is found from eq. (2.20) and can be written as

$$E_{\rm gr} = -\alpha \frac{GM^2}{R}, \tag{2.28}$$

where α is a constant of order unity (determined by the distribution of matter in the star, i.e. by the density profile). Using eq. (2.26), the internal energy of the star is $E_{\rm int} = \frac{3}{2}k/(\mu m_{\rm u}) \int T {\rm d}m = \frac{3}{2}k/(\mu m_{\rm u})\bar{T}M$, where $\bar{T}$ is the temperature averaged over all mass shells. By the virial theorem we then obtain

$$\bar{T} = \frac{\alpha}{3}\frac{\mu m_{\rm u}}{k}\frac{GM}{R}. \tag{2.29}$$

Taking $\alpha \approx 1$ and $\mu = 0.5$ for ionized hydrogen, we obtain for the Sun $\bar{T} \sim 4 \times 10^6$ K. This is the average temperature required to provide the pressure that is needed to keep the Sun in hydrostatic equilibrium. Since the temperature in a star normally decreases outwards, it is also an approximate lower limit on the central temperature of the Sun. At these temperatures, hydrogen and helium are indeed completely ionized. We shall see that $T_c \approx 10^7$ K is high enough for hydrogen fusion to take place in the central regions of the Sun.

The virial theorem for a general equation of state Also for equations of state other than an ideal gas a relation between pressure and internal energy exists, which we can write generally as

$$u = \phi \frac{P}{\rho}. \tag{2.30}$$

We have seen above that $\phi = \frac{3}{2}$ for an ideal gas, but it will turn out (see Ch. 3) that this is valid not only for an ideal gas, but for all non-relativistic particles. On the other hand, if we consider a gas of relativistic particles, in particular photons (i.e. radiation pressure), $\phi = 3$. If ϕ is constant throughout the star we can integrate the left-hand side of eq. (2.23) to obtain a more general form of the virial theorem:

$$E_{\rm int} = -\tfrac{1}{3}\phi E_{\rm gr} \tag{2.31}$$

2.3.1 The total energy of a star

The total energy of a star is the sum of its gravitational potential energy, its internal energy and its kinetic energy $E_{\rm kin}$ (due to bulk motions of gas inside the star, not the thermal motions of the gas particles):

$$E_{\rm tot} = E_{\rm gr} + E_{\rm int} + E_{\rm kin}. \tag{2.32}$$

The star is bound as long as its total energy is negative.

For a star in hydrostatic equilibrium we can set $E_{\rm kin} = 0$. Furthermore for a star in HE the virial theorem holds, so that $E_{\rm gr}$ and $E_{\rm int}$ are tightly related by eq. (2.31). Combining eqs. (2.31) and (2.32) we obtain the following relations:

$$E_{\rm tot} \;=\; E_{\rm int} + E_{\rm gr} \;=\; \frac{\phi - 3}{\phi} E_{\rm int} \;=\; (1 - \tfrac{1}{3}\phi) E_{\rm gr} \tag{2.33}$$

As long as $\phi < 3$ the star is bound. This is true in particular for the important case of a star consisting of an ideal gas (eq. 2.27), for which we obtain

$$\boxed{E_{\rm tot} \;=\; E_{\rm int} + E_{\rm gr} \;=\; -E_{\rm int} \;=\; \tfrac{1}{2}E_{\rm gr} \;<\; 0} \tag{2.34}$$

In other words, its total energy of such a star equals half of its gravitational potential energy.

From eq. (2.34) we can see that the virial theorem has the following important consequences:

- Gravitationally bound gas spheres must be *hot* to maintain hydrostatic equilibrium: heat provides the pressure required to balance gravity. The more compact such a sphere, the more strongly bound, and therefore the hotter it must be.

- A hot sphere of gas radiates into surrounding space, therefore a star must lose energy from its surface. The rate at which energy is radiated from the surface is the *luminosity* of the star. In the absence of an internal energy source, this energy loss must equal the decrease of the total energy of the star: $L = -\mathrm{d}E_{\rm tot}/\mathrm{d}t > 0$, since L is positive by convention.

- Taking the time derivative of eq. (2.34), we find that as a consequence of losing energy:

  $$\dot{E}_{\rm gr} = -2L < 0,$$

  meaning that the star *contracts* (becomes more strongly bound), and

  $$\dot{E}_{\rm int} = L > 0,$$

  meaning that the star *gets hotter* – unlike familiar objects which cool when they lose energy. Therefore a star can be said to have a *negative heat capacity*. Half the energy liberated by contraction is used for heating the star, the other half is radiated away.

For the case of a star that is dominated by radiation pressure, we find that $E_{\rm int} = -E_{\rm gr}$, and therefore the total energy $E_{\rm tot} = 0$. Therefore a star dominated by radiation pressure (or more generally, by the pressure of relativistic particles) is only marginally bound. No energy is required to expand or contract such a star, and a small perturbation would be enough to render it unstable and to trigger its collapse or complete dispersion.

2.3.2 Thermal equilibrium

If internal energy sources are present in a star due to nuclear reactions taking place in the interior, then the energy loss from the surface can be compensated: $L = L_{\rm nuc} \equiv -\mathrm{d}E_{\rm nuc}/\mathrm{d}t$. In that case the total energy is conserved and eq. (2.34) tells us that $\dot{E}_{\rm tot} = \dot{E}_{\rm int} = \dot{E}_{\rm gr} = 0$. The virial theorem therefore states that both $E_{\rm int}$ and $E_{\rm gr}$ are conserved as well: the star cannot, for example, contract and cool while keeping its total energy constant.

In this state, known as *thermal equilibrium* (TE), the star is in a stationary state. Energy is radiated away at the surface at the same rate at which it is produced by nuclear reactions in the interior. The star neither expands nor contracts, and it maintains a constant interior temperature. We shall see later that this temperature is regulated by the nuclear reactions themselves, which in combination with the virial theorem act like a stellar thermostat. Main-sequence stars like the Sun are in thermal equilibrium, and a star can remain in this state as long as nuclear reactions can supply the necessary energy.

Note that the arguments given above imply that both hydrostatic equilibrium and thermal equilibrium are *stable* equilibria, an assumption that we have yet to prove (see Ch. 7). It is relatively easy to understand why TE is stable, at least as long as the ideal-gas pressure dominates ($\phi < 3$ in eq. 2.31). Consider what happens when TE is disturbed, e.g. when $L_{\rm nuc} > L$ temporarily. The total energy then increases, and the virial theorem states that as a consequence the star must expand and cool. Since the nuclear reaction rates typically increase strongly with temperature, the rate of nuclear burning and thus $L_{\rm nuc}$ will decrease as a result of this cooling, until TE is restored when $L = L_{\rm nuc}$.

2.4 The timescales of stellar evolution

Three important timescales are relevant for stellar evolution, associated with changes to the mechanical structure of a star (described by the equation of motion, eq. 2.11), changes to its thermal structure (as follows from the virial theorem, see also Sect. 5.1) and changes in its composition, which will be discussed in Ch. 6.

The first timescale was already treated in Sec. 2.2.1: it is the *dynamical timescale* given by eq. (2.18),

$$\tau_{\rm dyn} \approx \sqrt{\frac{R^3}{GM}} \approx 0.02 \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M_\odot}{M}\right)^{1/2} \text{days} \tag{2.35}$$

The dynamical timescale is the timescale on which a star reacts to a perturbation of hydrostatic equilibrium. We saw that this timescale is typically of the order of hours or less, which means that stars are extremely close to hydrostatic equilibrium.

2.4.1 The thermal timescale

The second timescale describes how fast changes in the thermal structure of a star can occur. It is therefore also the timescale on which a star in thermal equilibrium reacts when its TE is perturbed. To obtain an estimate, we turn to the virial theorem: we saw in Sec. 2.3.1 that a star without a nuclear energy source contracts by radiating away its internal energy content: $L = \dot{E}_{\rm int} \approx -2\dot{E}_{\rm gr}$, where the last equality applies strictly only for an ideal gas. We can thus define the *thermal* or *Kelvin-Helmholtz timescale* as the timescale on which this gravitational contraction would occur:

$$\tau_{\rm KH} = \frac{E_{\rm int}}{L} \approx \frac{|E_{\rm gr}|}{2L} \approx \frac{GM^2}{2RL} \approx 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \frac{R_\odot}{R} \frac{L_\odot}{L} \text{ yr} \tag{2.36}$$

Here we have used eq. (2.28) for $E_{\rm gr}$ with $\alpha \approx 1$.

The thermal timescale for the Sun is about 1.5×10^7 years, which is many orders of magnitude larger than the dynamical timescale. There is therefore no direct observational evidence that any star is in thermal equilibrium. In the late 19th century gravitational contraction was proposed as the energy source of the Sun by Lord Kelvin and, independently, by Hermann von Helmholtz. This led to an age of the Sun and an upper limit to the age the Earth that was in conflict with emerging geological evidence, which required the Earth to be much older. Nuclear reactions have since turned out to be a much more powerful energy source than gravitational contraction, allowing stars to be in thermal equilibrium for most (> 99 %) of their lifetimes. However, several phases of stellar evolution, during which the nuclear power source is absent or inefficient, do occur on the thermal timescale.

2.4.2 The nuclear timescale

A star can remain in thermal equilibrium for as long as its nuclear fuel supply lasts. The associated timescale is called the *nuclear timescale*, and since nuclear fuel (say hydrogen) is burned into 'ash' (say helium), it is also the timescale on which composition changes in the stellar interior occur.

The energy source of nuclear fusion is the direct conversion of a small fraction ϕ of the rest mass of the reacting nuclei into energy. For hydrogen fusion, $\phi \approx 0.007$; for fusion of helium and heavier elements ϕ is smaller by a factor 10 or more. The total nuclear energy supply can therefore be written as $E_{\rm nuc} = \phi M_{\rm nuc} c^2 = \phi f_{\rm nuc} M c^2$, where $f_{\rm nuc}$ is that fraction of the mass of the star which may serve as nuclear fuel. In thermal equilibrium $L = L_{\rm nuc} = \dot{E}_{\rm nuc}$, so we can estimate the nuclear timescale as

$$\tau_{\rm nuc} = \frac{E_{\rm nuc}}{L} = \phi f_{\rm nuc} \frac{Mc^2}{L} \approx 10^{10} \frac{M}{M_\odot} \frac{L_\odot}{L} \text{ yr.} \tag{2.37}$$

The last approximate equality holds for hydrogen fusion in a star like the Sun, with has 70 % of its initial mass in hydrogen and fusion occurring only in the inner $\approx$ 10 % of its mass (the latter result comes from detailed stellar models). This long timescale is consistent with the geological evidence for the age of the Earth.

We see that, despite only a small fraction of the mass being available for fusion, the nuclear timescale is indeed two to three orders of magnitude larger than the thermal timescale. Therefore the assumption that stars can reach a state of thermal equilibrium is justified. To summarize, we have found:

$$\tau_{\rm nuc} \gg \tau_{\rm KH} \gg \tau_{\rm dyn}.$$

As a consequence, the rates of nuclear reactions determine the pace of stellar evolution, and stars may be assumed to be in hydrostatic and thermal equilibrium throughout most of their lives.

Suggestions for further reading

The contents of this chapter are covered more extensively by Chapter 1 of Maeder and by Chapters 1 to 4 of Kippenhahn & Weigert.

Exercises

2.1 Density profile

In a star with mass M, assume that the density decreases from the center to the surface as a function of radial distance r, according to

$$\rho = \rho_c \left[1 - \left(\frac{r}{R} \right)^2 \right], \tag{2.38}$$

where ρ_c is a given constant and R is the radius of the star.

(a) Find $m(r)$.

(b) Derive the relation between M and R.

(c) Show that the average density of the star is $0.4\rho_c$.

2.2 Hydrostatic equilibrium

(a) Consider an infinitesimal mass element dm inside a star, see Fig. 2.1. What forces act on this mass element?

(b) Newton's second law of mechanics, or the equation of motion, states that the net force acting on a body is equal to its acceleration times it mass. Write down the equation of motion for the gas element.

(c) In hydrostatic equilibrium the net force is zero and the gas element is not accelerated. Find an expression of the pressure gradient in hydrostatic equilibrium.

(d) Find an expression for the central pressure P_c by integrating the pressure gradient. Use this to derive the lower limit on the central pressure of a star in hydrostatic equilibrium, eq. (2.16).

(e) Verify the validity of this lower limit for the case of a star with the density profile of eq. (2.38).

2.3 The virial theorem

An important consequence of hydrostatic equilibrium is that it links the gravitational potential energy $E_{\rm gr}$ and the internal thermal energy $E_{\rm int}$.

(a) Estimate the gravitational energy $E_{\rm gr}$ for a star with mass M and radius R, assuming (1) a constant density distribution and (2) the density distribution of eq. (2.38).

(b) Assume that a star is made of an ideal gas. What is the kinetic internal energy per particle for an ideal gas? Show that the total internal energy, $E_{\rm int}$ is given by:

$$E_{\rm int} = \int_0^R \left(\frac{3}{2} \frac{k}{\mu m_{\rm u}} \rho(r) T(r) \right) 4\pi r^2 \, {\rm d}r. \tag{2.39}$$

(c) Estimate the internal energy of the Sun by assuming constant density and $T(r) \approx \langle T \rangle \approx \frac{1}{2} T_{\rm c} \approx 5 \times 10^6 K$ and compare your answer to your answer for a). What is the total energy of the Sun? Is the Sun bound according to your estimates?

It is no coincidence that the order of magnitude for $E_{\rm gr}$ and $E_{\rm int}$ are the same[1]. This follows from hydrostatic equilibrium and the relation is known as the virial theorem. In the next steps we will derive the virial theorem starting from the pressure gradient in the form of eq. (2.12).

(d) Multiply by both sides of eq. (2.12) by $4\pi r^3$ and integrate over the whole star. Use integration by parts to show that

$$\int_0^R 3\, P \; 4\pi r^2 \, {\rm d}r = \int_0^R \frac{G m(r)}{r} \rho 4\pi r^2 \, {\rm d}r. \tag{2.40}$$

(e) Now derive a relation between $E_{\rm gr}$ and $E_{\rm int}$, the virial theorem for an ideal gas.

(f) (*) Also show that for the average pressure of the star

$$\langle P \rangle = \frac{1}{V} \int_0^{R_*} P \; 4\pi r^2 \, {\rm d}r = -\frac{1}{3} \frac{E_{\rm gr}}{V}, \tag{2.41}$$

where V is the volume of the star.

As the Sun evolved towards the main sequence, it contracted under gravity while remaining close to hydrostatic equilibrium. Its internal temperature changed from about 30 000 K to about 6×10^6K.

(g) Find the total energy radiated during away this contraction. Assume that the luminosity during this contraction is comparable to $L_\odot$ and estimate the time taken to reach the main sequence.

2.4 Conceptual questions

[1] In reality $E_{\rm gr}$ is larger than estimated above because the mass distribution is more concentrated to the centre.

(a) Use the virial theorem to explain why stars are hot, i.e. have a high internal temperature and therefore radiate energy.

(b) What are the consequences of energy loss for the star, especially for its temperature?

(c) Most stars are in thermal equilibrium. What is compensating for the energy loss?

(d) What happens to a star in thermal equilibrium (and in hydrostatic equilibrium) if the energy production by nuclear reactions in a star drops (slowly enough to maintain hydrostatic equilibrium)?

(e) Why does this have a stabilizing effect? On what time scale does the change take place?

(f) What happens if hydrostatic equilibrium is violated, e.g. by a sudden increase of the pressure.

(g) On which timescale does the change take place? Can you give examples of processes in stars that take place on this timescale.

2.5 Three important timescales in stellar evolution

(a) The nuclear timescale τ_{nuc}.

i. Calculate the total mass of hydrogen available for fusion over the lifetime of the Sun, if 70% of its mass was hydrogen when the Sun was formed, and only 13% of all hydrogen is in the layers where the temperature is high enough for fusion.

ii. Calculate the fractional amount of mass converted into energy by hydrogen fusion. (Refer to Table 1 for the mass of a proton and of a helium nucleus.)

iii. Derive an expression for the nuclear timescale in solar units, i.e. expressed in terms of $R/R_\odot$, $M/M_\odot$ and $L/L_\odot$.

iv. Use the mass-radius and mass-luminosity relations for main-sequence stars to express the nuclear timescale of main-sequence stars as a function of the mass of the star only.

v. Describe in your own words the meaning of the nuclear timescale.

(b) The thermal timescale τ_{KH}.

i-iii. Answer question (a) iii, iv and v for the thermal timescale and calculate the age of the Sun according to Kelvin.

iv. Why are most stars observed to be main-sequence stars and why is the Hertzsprung-gap called a gap?

(c) The dynamical timescale τ_{dyn}.

i-iii. Answer question (a) iii, iv and v for the dynamical timescale.

iv. In stellar evolution models one often assumes that stars evolve *quasi-statically*, i.e. that the star remains in hydrostatic equilibrium throughout. Why can we make this assumption?

v. Rapid changes that are sometimes observed in stars may indicate that dynamical processes are taking place. From the timescales of such changes - usually oscillations with a characteristic period - we may roughly estimate the average density of the Star. The sun has been observed to oscillate with a period of minutes, white dwarfs with periods of a few tens of seconds. Estimate the average density for the Sun and for white dwarfs.

(d) Comparison.

i. Summarize your results for the questions above by computing the nuclear, thermal and dynamical timescales for a 1, 10 and 25 $M_\odot$ main-sequence star. Put your answers in tabular form.

ii. For each of the following evolutionary stages indicate on which timescale they occur: *pre-main sequence contraction, supernova explosion, core hydrogen burning, core helium burning*.

iii. When the Sun becomes a red giant (RG), its radius will increase to $200R_\odot$ and its luminosity to $3000L_\odot$. Estimate τ_{dyn} and τ_{KH} for such a RG.

iv. How large would such a RG have to become for $\tau_{\text{dyn}} > \tau_{\text{KH}}$? Assume both R and L increase at constant effective temperature.