Chapter 3

Equation of state of stellar interiors

3.1 Local thermodynamic equilibrium

Empirical evidence shows that in a part of space isolated from the rest of the Universe, matter and radiation tend towards a state of *thermodynamic equilibrium*. This equilibrium state is achieved when sufficient interactions take place between the material particles ('collisions') and between the photons and mass particles (scatterings and absorptions). In such a state of thermodynamic equilibrium the radiation field becomes isotropic and the photon energy distribution is described by the Planck function (blackbody radiation). The statistical distribution functions of both the mass particles and the photons are then characterized by a single temperature T.

We know that stars are not isolated systems, because they emit radiation and generate (nuclear) energy in their interiors. Indeed, the surface temperature of the Sun is about 6000 K, while we have estimated from the virial theorem (Sec. 2.3) that the interior temperature must of the order of 10^7 K. Therefore stars are *not* in global thermodynamic equilibrium. However, it turns out that locally within a star, a state of thermodynamic equilibrium *is* achieved. This means that within a region much smaller than the dimensions of a star ($\ll R_*$), but larger than the average distance between interactions of the particles (both gas particles and photons), i.e. larger than the mean free path, there is a well-defined *local temperature* that describes the particle statistical distributions.

We can make this plausible by considering the mean free path for photons:

$$\ell_{\rm ph} = 1/\kappa\rho$$

where κ is the opacity coefficient, i.e. the effective cross section per unit mass. For fully ionized matter, a minimum is given by the electron scattering cross section, which is $\kappa_{es} = 0.4 \text{ cm}^2/\text{g}$ (see Ch. 5). The average density in the Sun is $\bar{\rho} = 1.4 \text{ g/cm}^3$, which gives a mean free path of the order of $\ell_{ph} \sim 1 \text{ cm}$. In other words, stellar matter is very opaque to radiation. The temperature difference over a distance ℓ_{ph} , i.e. between emission and absorption, can be estimated as

$$\Delta T \approx \frac{dT}{dr} \ell_{\rm ph} \approx \frac{T_c}{R} \ell_{\rm ph} \approx \frac{10^7}{10^{11}} \approx 10^{-4} \, {\rm K}$$

which is a tiny fraction (10^{-11}) of the typical interior temperature of 10^7 K. Using a similar estimate, it can be shown that the mean free path for interactions between ionized gas particles (ions and electrons) is several orders of magnitude smaller than $\ell_{\rm ph}$. Hence a small region can be defined (a 'point' for all practical purposes) which is > $\ell_{\rm ph}$ but much smaller than the length scale over which significant changes of thermodynamic quantities occur. This is called *local thermodynamic equilibrium* (LTE). We can therefore assume a well-defined temperature distribution inside the star. Furthermore, the average time between particle interactions (the mean free time) is much shorter than the timescale for changes of the macroscopic properties. Therefore a state of LTE is secured at all times in the stellar interior. The assumption of LTE¹ constitutes a great simplification. It enables the calculation of all thermodynamic properties of the stellar gas in terms of the local values of temperature, density and composition, as they change from the centre to the surface.

3.2 The equation of state

The equation of state (EOS) describes the microscopic properties of stellar matter, for given density ρ , temperature *T* and composition X_i . It is usually expressed as the relation between the pressure and these quantities:

$$P = P(\rho, T, X_i) \tag{3.1}$$

Using the laws of thermodynamics, and a similar equation for the internal energy $U(\rho, T, X_i)$, we can derive from the EOS the thermodynamic properties that are needed to describe the structure of a star, such as the specific heats c_V and c_P , the adiabatic exponent γ_{ad} and the adiabatic temperature gradient ∇_{ad} .

An example is the ideal-gas equation of state, which in the previous chapters we have tacitly assumed to hold for stars like the Sun:

$$P = nkT$$
 or $P = \frac{k}{\mu m_{\rm u}}\rho T$.

In this chapter we will see whether this assumption was justified, and how the EOS can be extended to cover all physical conditions that may prevail inside a star. The ideal-gas law pertains to particles that behave according to classical physics. However, both quantum-mechanical and special relativistic effects may be important under the extreme physical conditions in stellar interiors. In addition, photons (which can be described as extremely relativistic particles) can be an important source of pressure.

We can define an ideal or *perfect* gas as a mixture of free, non-interacting particles. Of course the particles in such a gas do interact, so more precisely we require that their interaction energies are small compared to their kinetic energies. In that case the internal energy of the gas is just the sum of all kinetic energies. From statistical mechanics we can derive the properties of such a perfect gas, both in the classical limit (recovering the ideal-gas law) and in the quantum-mechanical limit (leading to electron degeneracy), and both in the non-relativistic and in the relativistic limit (e.g. valid for radiation). This is done in Sect. 3.3.

In addition, various *non-ideal* effects may become important. The high temperatures (> 10^6 K) in stellar interiors ensure that the gas will be fully ionized, but at lower temperatures (in the outer layers) partial ionization has to be considered, with important effects on the thermodynamic properties (see Sect. 3.5). Furthermore, in an ionized gas *electrostatic interactions* between the ions and electrons may be important under certain circumstances (Sect. 3.6).

3.3 Equation of state for a gas of free particles

We shall derive the equation of state for a perfect gas from the principles of statistical mechanics. This provides a description of the ions, the electrons, as well as the photons in the deep stellar interior.

¹N.B. note the difference between (local) *thermodynamic equilibrium* ($T_{gas}(r) = T_{rad}(r) = T(r)$) and the earlier defined, global property of *thermal equilibrium* ($E_{tot} = const$, or $L = L_{nuc}$).

Let n(p) be the distribution of momenta of the gas particles, i.e. n(p) dp represents the number of particles per unit volume with momenta $p \in [p \dots p + dp]$. If n(p) is known then the total number density (number of particles per unit volume), the internal energy density (internal energy per unit volume) and the pressure can be obtained from the following integrals:

number density
$$n = \int_0^\infty n(p) \, \mathrm{d}p$$
 (3.2)

internal energy density $U = \int_0^\infty \epsilon_p n(p) \, dp = n \langle \epsilon_p \rangle$

pressure
$$P = \frac{1}{3} \int_0^\infty p v_p n(p) \, \mathrm{d}p = \frac{1}{3} n \langle p v_p \rangle$$
 (3.4)

(3.3)

Here ϵ_p is the kinetic energy of a particle with momentum *p*, and v_p is its velocity. Eq. (3.2) is trivial, and eq. (3.3) follows from the perfect-gas assumption. The pressure integral eq. (3.4) requires some explanation.

Consider a gas of *n* particles in a cubical box with sides of length L = 1 cm. Each particle bounces around in the box, and the pressure on one side of the box results from the momentum imparted by all the particles colliding with it. Consider a particle with momentum *p* and corresponding velocity *v* coming in at an angle θ with the normal to the surface, as depicted in Fig. 3.1. The time between two collisions with the same side is

$$\Delta t = \frac{2L}{v\cos\theta} = \frac{2}{v\cos\theta}.$$

The collisions are elastic, so the momentum transfer is twice the momentum component perpendicular to the surface,

$$\Delta p = 2p\cos\theta. \tag{3.5}$$

The momentum transferred per particle per second and per cm² is therefore

$$\frac{\Delta p}{\Delta t} = vp\,\cos^2\theta.\tag{3.6}$$

The number of particles in the box with $p \in [p \dots p + dp]$ and $\theta \in [\theta \dots \theta + d\theta]$ is denoted as $n(\theta, p) d\theta dp$. The contribution to the pressure from these particles is then

$$dP = vp \cos^2 \theta \, n(\theta, p) \, d\theta \, dp. \tag{3.7}$$



Figure 3.1. Gas particle in a cubical box with a volume of 1 cm^3 . Each collision with the side of the box results in a transfer of momentum; the pressure inside the box is the result of the collective momentum transfers of all *n* particles in the box.

Since the momenta are distributed isotropically over all directions within a solid angle 2π , and the solid angle $d\omega$ subtended by those particles with $\theta \in [\theta \dots \theta + d\theta]$ equals $2\pi \sin \theta \, d\theta$, we have $n(\theta, p) \, d\theta = n(p) \sin \theta \, d\theta$ and

$$dP = vp n(p) \cos^2 \theta \sin \theta \, d\theta \, dp. \tag{3.8}$$

The total pressure is obtained by integrating over all angles $(0 \le \theta \le \pi/2)$ and momenta. This results in eq. (3.4) since $\int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta = \int_0^1 \cos^2 \theta \, d \cos \theta = \frac{1}{3}$.

3.3.1 Relation between pressure and internal energy

In general, the particle energies and velocities are related to their momenta according to special relativity:

$$\epsilon^2 = p^2 c^2 + m^2 c^4, \qquad \epsilon_p = \epsilon - mc^2 \tag{3.9}$$

and

$$v_p = \frac{\partial \epsilon}{\partial p} = \frac{pc^2}{\epsilon}.$$
(3.10)

We can obtain generally valid relations between the pressure and the internal energy of a perfect gas in the non-relativistic (NR) limit and the extremely relativistic (ER) limit:

NR limit: in this case the momenta $p \ll mc$, so that $\epsilon_p = \epsilon - mc^2 = \frac{1}{2}p^2/m$ and v = p/m. Therefore $\langle pv \rangle = \langle p^2/m \rangle = 2\langle \epsilon_p \rangle$ so that eq. (3.4) yields

$$P = \frac{2}{3}U\tag{3.11}$$

ER limit: in this case $p \gg mc$, so that $\epsilon_p = pc$ and v = c. Therefore $\langle pv \rangle = \langle pc \rangle = \langle \epsilon_p \rangle$, and eq. (3.4) yields

$$P = \frac{1}{3}U\tag{3.12}$$

These relations are generally true, for *any particle* (electrons, ions and photons). We will apply this in the coming sections. As we saw in the previous Chapter, the change from $\frac{2}{3}$ to $\frac{1}{3}$ in the relation has important consequences for the virial theorem, and for the stability of stars.

3.3.2 The classical ideal gas

Using the tools of statistical mechanics, we can address the origin of the ideal-gas law. The momentum distribution n(p) for classical, non-relativistic particles of mass m in LTE is given by the Maxwell-Boltzmann distribution:

$$n(p) dp = \frac{n}{(2\pi m kT)^{3/2}} e^{-p^2/2m kT} 4\pi p^2 dp.$$
(3.13)

Here the exponential factor $(e^{-\epsilon_p/kT})$ represents the equilibrium distribution of kinetic energies, the factor $4\pi p^2 dp$ is the volume in momentum space (p_x, p_y, p_z) for $p \in [p \dots p + dp]$, and the factor $n/(2\pi mkT)^{3/2}$ comes from the normalization of the total number density *n* imposed by eq. (3.2). (You can verify this by starting from the standard integral $\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi/a}$, and differentiating once with respect to *a* to obtain the integral $\int_0^\infty e^{-ax^2} x^2 dx$.)

The pressure is calculated by using v = p/m for the velocity in eq. (3.4):

$$P = \frac{1}{3} \frac{n}{(2\pi m kT)^{3/2}} \int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} 4\pi p^2 dp.$$
(3.14)

By performing the integration (for this you need to differentiate $\int_0^\infty e^{-ax^2} x^2 dx$ once more with respect to *a*) you can verify that this indeed yields the ideal gas law

$$P = nkT (3.15)$$

(N.B. This derivation is for a gas of *non-relativistic* classical particles, but it can be shown that the same relation P = nkT is also valid for *relativistic* classical particles.)

3.3.3 Mixture of ideal gases, and the mean molecular weight

The ideal gas relation was derived for identical particles of mass m. It should be obvious that for a mixture of free particles of different species, it holds for the partial pressures of each of the constituents of the gas separately. In particular, it holds for both the ions and the electrons, as long as quantum-mechanical effects can be ignored. The total gas pressure is then just the sum of partial pressures

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}} = \sum_{i} P_{i} + P_{\text{e}} = (\sum_{i} n_{i} + n_{\text{e}})kT = nkT$$

where n_i is the number density of ions of element *i*, with mass $m_i = A_i m_u$ and charge $Z_i e$. Then n_i is related to the density and the mass fraction X_i of this element as

$$n_i = \frac{X_i \rho}{A_i m_{\rm u}}$$
 and $n_{\rm ion} = \sum_i \frac{X_i}{A_i} \frac{\rho}{m_{\rm u}} \equiv \frac{1}{\mu_{\rm ion}} \frac{\rho}{m_{\rm u}},$ (3.16)

which defines the mean atomic mass per ion μ_{ion} . The partial pressure due to all ions is then

$$P_{\rm ion} = \frac{1}{\mu_{\rm ion}} \frac{\rho}{m_{\rm u}} kT = \frac{\mathcal{R}}{\mu_{\rm ion}} \rho T.$$
(3.17)

We have used here the universal gas constant $\mathcal{R} = k/m_u = 8.31447 \times 10^7 \text{ erg g}^{-1} \text{ K}^{-1}$. The number density of electrons is given by

$$n_{\rm e} = \sum_{i} Z_i n_i = \sum_{i} \frac{Z_i X_i}{A_i} \frac{\rho}{m_{\rm u}} \equiv \frac{1}{\mu_{\rm e}} \frac{\rho}{m_{\rm u}},\tag{3.18}$$

which defines the *mean molecular weight per free electron* μ_e . As long as the electrons behave like classical particles, the electron pressure is thus given by

$$P_{\rm e} = \frac{1}{\mu_{\rm e}} \frac{\rho}{m_{\rm u}} kT = \frac{\mathcal{R}}{\mu_{\rm e}} \rho T.$$
(3.19)

When the gas is fully ionized, we have for hydrogen $Z_i = A_i = 1$ while for helium and the most abundant heavier elements, $Z_i/A_i \approx \frac{1}{2}$. In terms of the hydrogen mass fraction X we then get

$$\mu_{\rm e} \approx \frac{2}{1+X},\tag{3.20}$$

which for the Sun (X = 0.7) amounts to $\mu_e \approx 1.18$, and for hydrogen-depleted gas gives $\mu_e \approx 2$.

The total gas pressure is then given by

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}} = \left(\frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}}\right) \mathcal{R}\rho T = \frac{\mathcal{R}}{\mu}\rho T$$
(3.21)

where the *mean molecular weight* μ is given by

$$\frac{1}{\mu} = \frac{1}{\mu_{\rm ion}} + \frac{1}{\mu_{\rm e}} = \sum_{i} \frac{(Z_i + 1)X_i}{A_i}.$$
(3.22)

It is left as an exercise to show that for a fully ionized gas, μ can be expressed in terms of the mass fractions X, Y and Z as

$$\mu \approx \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}$$
(3.23)

if we assume that for elements heavier than helium, $A_i \approx 2Z_i \approx 2(Z_i + 1)$.

3.3.4 Quantum-mechanical description of the gas

According to quantum mechanics, the accuracy with which a particle's location and momentum can be known simultaneously is limited by Heisenberg's uncertainty principle, i.e. $\Delta x \Delta p \ge h$. In three dimensions, this means that if a particle is located within a volume element ΔV then its localization within three-dimensional momentum space $\Delta^3 p$ is constrained by

$$\Delta V \,\Delta^3 p \ge h^3. \tag{3.24}$$

The quantity h^3 defines the volume in six-dimensional phase space of one quantum cell. The *number* of quantum states in a spatial volume V and with momenta $p \in [p \dots p + dp]$ is therefore given by

$$g(p) dp = g_s \frac{V}{h^3} 4\pi p^2 dp,$$
(3.25)

where g_s is the number of intrinsic quantum states of the particle, e.g. spin or polarization.

The relative occupation of the available quantum states for particles in thermodynamic equilibrium depends on the type of particle:

• *fermions* (e.g. electrons or nucleons) obey the Pauli exclusion principle, which postulates that no two such particles can occupy the same quantum state. The fraction of states with energy ϵ_p that will be occupied at temperature T is given by

$$f_{\rm FD}(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} + 1},$$
 (3.26)

which is always ≤ 1 .

• *bosons* (e.g. photons) have no restriction on the number of particles per quantum state, and the fraction of states with energy ϵ_p that is occupied is

$$f_{\rm BE}(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} - 1},$$
 (3.27)

which can be > 1.

The actual distribution of momenta for particles in LTE is given by the product of the occupation fraction $f(\epsilon_p)$ and the number of quantum states, given by eq. (3.25). The quantity μ appearing in eqs. (3.26) and (3.27) is the so-called *chemical potential*. It can be seen as a normalization constant, determined by the total number of particles in the volume considered (i.e., by the constraint imposed by eq. 3.2).



Figure 3.2. Left: Electron momentum distributions n(p) for an electron density of $n_e = 6 \times 10^{27}$ cm⁻³ (corresponding to $\rho = 2 \times 10^4$ g/cm⁻³ if $\mu_e = 2$), and for three different temperatures: $T = 2 \times 10^7$ K (black lines), 2×10^6 K (red lines) and 2×10^5 K (blue lines). The actual distributions, governed by quantum mechanics, are shown as solid lines while the Maxwell-Boltzmann distributions for the same n_e and T values are shown as dashed lines. The dotted line n_{max} is the maximum possible number distribution if all quantum states with momentum p are occupied. Right: Distributions in the limit T = 0, when all lowest available momenta are fully occupied. The blue line is for the same density as in the left panel, while the red line is for a density two times as high.

3.3.5 Electron degeneracy

Electrons are fermions with two spin states, i.e. $g_e = 2$. According to eq. (3.25), the maximum number density of electrons with momentum p allowed by quantum mechanics is therefore

$$n_{\max}(p) dp = \frac{g_e}{h^3} 4\pi p^2 dp = \frac{8\pi}{h^3} p^2 dp.$$
(3.28)

This is shown as the dotted line in Fig. 3.2. The actual momentum distribution of electrons $n_e(p)$ is given by the product of eq. (3.28) and eq. (3.26). In the non-relativistic limit we have $\epsilon_p = p^2/2m_e$, giving

$$n_{\rm e}(p)\,{\rm d}p = \frac{2}{h^3} \frac{1}{e^{(p^2/2m_{\rm e}kT) - \psi} + 1} \,4\pi p^2\,{\rm d}p, \tag{3.29}$$

where we have replaced the chemical potential by the *degeneracy parameter* $\psi = \mu/kT$. The value of ψ is determined by the constraint that $\int_0^\infty n_e(p) dp = n_e$ (eq. 3.2).

The limitation imposed by the Pauli exclusion principle means that electrons can exert a higher pressure than predicted by classical physics (eq. 3.19). To illustrate this, in Fig. 3.2 the momentum distribution eq. (3.29) is compared to the Maxwell-Boltzmann distribution for electrons, eq. (3.13),

$$n_{\rm MB}(p)\,\mathrm{d}p = \frac{n_{\rm e}}{(2\pi m_{\rm e} kT)^{3/2}} \,e^{-p^2/2m_{\rm e} kT} \,4\pi p^2\,\mathrm{d}p. \tag{3.30}$$

The situation shown is for an electron density $n_e = 6 \times 10^{27} \text{ cm}^{-3}$, which corresponds to a mass density of $2 \times 10^4 \text{ g/cm}^{-3}$ (assuming a hydrogen-depleted gas with $\mu_e = 2$). At high temperatures, $T = 2 \times 10^7 \text{ K}$, the momentum distribution (solid line) nearly coincides with the M-B distribution

(dashed line): none of the quantum states are fully occupied ($n_e(p) < n_{max}(p)$ for all values of p) and the electrons behave like classical particles. As the temperature is decreased, e.g. at $T = 2 \times 10^6$ K (red lines), the peak in the M-B distribution shifts to smaller p and is higher (since the integral over the distribution must equal n_e). The number of electrons with small values of p expected from classical physics, $n_{\rm MB}(p)$, then exceeds the maximum allowed by the Pauli exclusion principle, $n_{\rm max}(p)$. These electrons are forced to assume quantum states with higher p: the peak in the distribution $n_e(p)$ occurs at higher p. Due to the higher momenta and velocities of these electrons, the electron gas exerts a higher pressure than inferred from classical physics. This is called *degeneracy pressure*. If the temperature is decreased even more, e.g. at $T = 2 \times 10^5$ K (blue lines), the lowest momentum states become nearly all filled and $n_e(p)$ follows $n_{max}(p)$ until it drops sharply. In this state of strong degeneracy, further decrease of T hardly changes the momentum distribution, so that the electron pressure becomes nearly *independent of temperature*.

Complete electron degeneracy

In the limit that $T \to 0$, all available momentum states are occupied up to a maximum value, while all higher states are empty, as illustrated in the right panel of Fig. 3.2. This is known as *complete degeneracy*, and the maximum momentum is called the *Fermi momentum* $p_{\rm F}$. Then we have

$$n_{\rm e}(p) = \frac{8\pi p^2}{h^3}$$
 for $p \le p_{\rm F}$, (3.31)

$$n_{\rm e}(p) = 0$$
 for $p > p_{\rm F}$. (3.32)

The Fermi momentum is determined by the electron density through eq. (3.2), i.e. $\int_0^{p_F} n_e(p) dp = n_e$, which yields

$$p_{\rm F} = h \left(\frac{3}{8\pi} n_{\rm e}\right)^{1/3}.$$
(3.33)

The pressure of a completely degenerate electron gas is now easy to compute using the pressure integral eq. (3.4). It depends on whether the electrons are relativistic or not. In the *non-relativistic* limit we have v = p/m and hence

$$P_{\rm e} = \frac{1}{3} \int_0^{p_{\rm F}} \frac{8\pi p^4}{h^3 m_{\rm e}} \,\mathrm{d}p = \frac{8\pi}{15h^3 m_{\rm e}} \,p_{\rm F}^5 = \frac{h^2}{20m_{\rm e}} \left(\frac{3}{\pi}\right)^{2/3} n_{\rm e}^{5/3}.$$
(3.34)

Using eq. (3.18) for n_e this can be written as

$$P_{\rm e} = K_{\rm NR} \left(\frac{\rho}{\mu_{\rm e}}\right)^{5/3} \qquad \text{with} \qquad K_{\rm NR} = \frac{h^2}{20m_{\rm e} m_{\rm u}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} = 1.0036 \times 10^{13} \,[{\rm cgs}]. \tag{3.35}$$

As more electrons are squeezed into the same volume, they have to occupy states with larger momenta, as illustrated in Fig. 3.2. Therefore the electron pressure increases with density, as expressed by eq. (3.35).

If the electron density is increased further, at some point the velocity of the most energetic electrons, p_F/m_e , approaches the speed of light. We then have to replace v = p/m by the relativistic kinematics relation (3.10). In the *extremely relativistic* limit when the majority of electrons move at relativistic speeds, we can take v = c and

$$P_{\rm e} = \frac{1}{3} \int_0^{p_{\rm F}} \frac{8\pi c p^3}{h^3} \,\mathrm{d}p = \frac{8\pi c}{12h^3} \,p_{\rm F}^4 = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} n_{\rm e}^{4/3},\tag{3.36}$$



Figure 3.3. The equation of state for completely degenerate electrons. The slope of the log *P*-log ρ relation changes from 5/3 at relatively low densities, where the electrons are non-relativistic, to 4/3 at high density when the electrons are extremely relativistic. The transition is smooth, but takes place at densities around $\rho_{\rm tr} \approx 10^6 \mu_{\rm e} \,{\rm g \, cm^{-3}}$.

which gives

$$P_{\rm e} = K_{\rm ER} \left(\frac{\rho}{\mu_{\rm e}}\right)^{4/3} \qquad \text{with} \qquad K_{\rm ER} = \frac{hc}{8m_{\rm u}^{4/3}} \left(\frac{3}{\pi}\right)^{1/3} = 1.2435 \times 10^{15} \,[\rm cgs]. \tag{3.37}$$

In the ER limit the pressure still increases with density, but with a smaller exponent $(\frac{4}{3} \text{ instead of } \frac{5}{3})$. The transition between the NR regime, eq. (3.35), and the ER regime, eq. (3.37), is smooth and can be expressed as a function of $x = p_F/m_ec$, see MAEDER Sec. 7.7. Roughly, the transition occurs at a density ρ_{tr} given by the condition $p_F \approx m_ec$, which can be expressed as

$$\rho_{\rm tr} \approx \mu_{\rm e} \, m_{\rm u} \frac{8\pi}{3} \left(\frac{m_{\rm e} c}{h} \right)^3. \tag{3.38}$$

The relation between P_e and ρ for a completely degenerate electron gas is shown in Fig. 3.3.

Partial degeneracy

Although the situation of complete degeneracy is only achieved at T = 0, it is a very good approximation whenever the degeneracy is strong, i.e. when the temperature is sufficiently low, as illustrated by Fig. 3.2. It corresponds to the situation when the degeneracy parameter $\psi \gg 0$ in eq. (3.29). In that case eqs. (3.35) and (3.37) can still be used to calculate the pressure to good approximation.

The transition between the classical ideal gas situation and a state of strong degeneracy occurs smoothly, and is known as *partial degeneracy*. To calculate the pressure the full expression eq. (3.29) has to be used in the pressure integral, which becomes rather complicated. The integral then depends on ψ , and can be expressed as one of the so-called *Fermi-Dirac* integrals, see MAEDER Sec. 7.7 for details (the other Fermi-Dirac integral relates to the internal energy density U). The situation of partial degeneracy corresponds to $\psi \sim 0$.

When $\psi \ll 0$ the classical description is recovered, i.e. eq. (3.29) becomes the Maxwell-Boltzmann distribution. In that case $1/(e^{(p^2/2m_ckT)-\psi} + 1) = e^{-(p^2/2m_ckT)+\psi}$ and therefore

$$\frac{2}{h^3}e^{\psi} = \frac{n_{\rm e}}{(2\pi m_{\rm e} kT)^{3/2}} \qquad \text{or} \qquad \psi = \ln \frac{h^3 n_{\rm e}}{2(2\pi m_{\rm e} kT)^{3/2}}$$

This only holds for $\psi \ll 0$, but more generally it can be shown that $\psi = \psi(n_e/T^{2/3})$. We have to consider (partial) degeneracy if $\psi \gtrsim 0$, i.e. if

$$n_{\rm e} \gtrsim \frac{2(2\pi m_{\rm e} kT)^{3/2}}{h^3}.$$
 (3.39)

The limit of strong (almost complete) degeneracy is reached when n_e is roughly a factor 10 higher.

Importance of electron degeneracy in stars

As a star, or its core, contracts the density may become so high that the electrons become degenerate and exert a (much) higher pressure than they would if they behaved classically. Since in the limit of strong degeneracy the pressure no longer depends on the temperature, this degeneracy pressure can hold the star up against gravity, regardless of the temperature. Therefore a degenerate star does not have to be hot to be in hydrostatic equilibrium, and it can remain in this state forever even when it cools down. This is the situation in *white dwarfs*.

The importance of relativity is that, when a degenerate star becomes more compact and the density increases further, the pressure increases less steeply with density. This has important consequences for massive white dwarfs, and we shall see that it implies that there is a maximum mass for which white dwarfs can exist (the Chandrasekhar mass).

We note that although electron degeneracy can be (very) important in stars, degeneracy of the *ions* is not. Since the ions have masses $\gtrsim 2000$ larger than electrons, their momenta ($p = \sqrt{2m\epsilon}$) are much larger at energy equipartition, and the condition (3.39) above (with m_e replaced by m_{ion}) implies that much higher densities are required at a particular temperature. In practice this never occurs: before such densities are reached the protons in the atomic nuclei will capture free electrons, and the composition becomes one of (mostly) neutrons. Degeneracy of *neutrons* does become important when we consider neutron stars.

3.3.6 Radiation pressure

Photons can be treated as quantum-mechanical particles that carry momentum and therefore exert pressure when they interact with matter. In particular photons are *bosons* with $g_s = 2$ (two polarization states), so they can be described by the Bose-Einstein statistics, eq. (3.27). The number of photons is not conserved, they can be destroyed and created until thermodynamic equilibrium is achieved. This means that $\mu = 0$ in eq. (3.27) and hence

$$n(p) dp = \frac{2}{h^3} \frac{1}{e^{\epsilon_p/kT} - 1} 4\pi p^2 dp$$
(3.40)

Photons are completely relativistic with $\epsilon_p = pc = hv$, so in terms of frequency v their distribution in LTE becomes the *Planck function* for blackbody radiation:

$$n(\nu) \,\mathrm{d}\nu = \frac{8\pi}{c^3} \frac{\nu^2 \,\mathrm{d}\nu}{e^{h\nu/kT} - 1} \tag{3.41}$$

Applying eqs. (3.2) and (3.3) one can show that the photon number density and the energy density of radiation are

$$n_{\rm ph} = \int_0^\infty n(p) \,\mathrm{d}p = b \,T^3 \tag{3.42}$$

$$U_{\rm rad} = \int_0^\infty pc \, n(p) \, \mathrm{d}p = a \, T^4 \tag{3.43}$$

where $b = 20.3 \text{ cm}^{-3} \text{ K}^{-3}$ and *a* is the radiation constant

$$a = \frac{8\pi^5 k^4}{15h^3 c^3} = 7.56 \times 10^{-15} \,\mathrm{erg} \,\mathrm{cm}^{-3} \,\mathrm{K}^{-4}.$$

Since photons are always extremely relativistic, $P = \frac{1}{3}U$ by eq. (3.12) and the *radiation pressure* is given by

$$P_{\rm rad} = \frac{1}{3}aT^4 \tag{3.44}$$

Pressure of a mixture of gas and radiation

The pressure inside a star is the sum of the gas pressure and radiation pressure,

$$P = P_{\rm rad} + P_{\rm gas} = P_{\rm rad} + P_{\rm ion} + P_{\rm e}.$$

where P_{rad} is given by eq. (3.44) and P_{ion} by eq. (3.17). In general P_{e} must be calculated as described in Sect. 3.3.5. In the classical limit it is given by eq. (3.19), and in the limits of non-relativistic and extremely relativistic degeneracy by eqs. (3.35) and (3.37), respectively. If the electrons are nondegenerate then the pressure can be written as

$$P = \frac{1}{3}aT^4 + \frac{\mathcal{R}}{\mu}\rho T.$$
 (3.45)

If the electrons are strongly degenerate their pressure dominates over that of the (classical) ions, so in that case P_{ion} can be neglected in the total pressure.

The fraction of the pressure contributed by the gas is customarily expressed as β , i.e.

$$P_{\text{gas}} = \beta P \quad \text{and} \quad P_{\text{rad}} = (1 - \beta) P. \tag{3.46}$$

3.3.7 Equation of state regimes

The different sources of pressure we have discussed so far dominate the equation of state at different temperatures and densities. In Fig. 3.4 the boundaries between these regimes are plotted schematically in the log T, log ρ plane.

- The boundary between regions where radiation and ideal-gas pressure dominate is defined by $P_{\text{rad}} = P_{\text{gas}}$, giving $T/\rho^{1/3} = 3.2 \times 10^7 \mu^{-1/3}$ when T and ρ are expressed in cgs units. (Verify this by comparing eqs. 3.21 and 3.44.) This is a line with slope $\frac{1}{3}$ in the log T vs log ρ plane.
- Similarly, the boundary between the regions dominated by ideal-gas pressure and non-relativistic degenerate electron pressure can be defined by $P_{\text{gas,ideal}} = P_{e,\text{NR}}$ as given by eq. (3.35), giving $T/\rho^{2/3} = 1.21 \times 10^5 \mu \mu_e^{-5/3}$ (again with T and ρ in cgs units). This is a line with slope $\frac{2}{3}$ in the log T-log ρ plane.
- The approximate boundary between non-relativistic and relativistic degeneracy is given by eq. (3.38), $\rho = 9.7 \times 10^5 \mu_e \text{ g/cm}^3$.
- At high densities the boundary between ideal gas pressure and extremely relativistic degeneracy is found by equating eqs. (3.21) and (3.37), giving $T/\rho^{1/3} = 1.50 \times 10^7 \mu \mu_e^{-4/3}$ (with T and ρ in cgs units), again a line with slope $\frac{1}{2}$.

As shown in Fig. 3.4, detailed models of zero-age (that is, homogeneous) main-sequence stars with masses between 0.1 and $100 M_{\odot}$ cover the region where ideal-gas pressure dominates the equation of state. This justifies the assumptions made in Ch. 2 when discussing the virial theorem and its consequences for stars, and when estimating temperatures in the stellar interior.



Figure 3.4. Left: The equation of state for a gas of free particles in the log T, log ρ plane. The dashed lines are approximate boundaries between regions where radiation pressure, ideal gas pressure, non-relativistic electron degeneracy and extremely relativistic electron degeneracy dominate, for a composition X = 0.7 and Z = 0.02. Right: Detailed structure models for homogeneous main-sequence stars of $0.1...100 M_{\odot}$ have been added (solid lines). The $1 M_{\odot}$ model is well within the ideal-gas region of the equation of state. In the $0.1 M_{\odot}$ star electron degeneracy pressure is important, except in the outer layers (at low ρ and T). In stars more massive than $10 M_{\odot}$, radiation pressure becomes important, and it dominates in the surface layers of the $100 M_{\odot}$ model.

3.4 Adiabatic processes

It is often important to consider processes that occur on such a short (e.g. hydrodynamical) timescale that there is no heat exchange with the environment; such processes are *adiabatic*. To derive the properties of stellar interiors under adiabatic conditions we need several thermodynamic derivatives. We therefore start from the laws of thermodynamics.

The *first law* of thermodynamics states that the amount of heat absorbed by a system (δQ) is the sum of the change in its internal energy (δU) and the work done on the system ($\delta W = P\delta V$). The *second law* of thermodynamics states that, for a reversible process, the change in entropy equals the change in the heat content divided by the temperature. Entropy is a state variable, unlike the heat content. For a unit mass (1 gram) of matter the combination of these laws can be expressed as

$$dq = T ds = du + P dv = du - \frac{P}{\rho^2} d\rho.$$
(3.47)

Here dq is the change in heat content, du is the change in internal energy ($u = U/\rho$ is the *specific* internal energy, i.e. per gram), s is the specific entropy (i.e. the entropy per unit mass) and $v = 1/\rho$ is the volume of a unit mass. Note that du and ds are exact differentials, whereas dq is not.

Differential form of the equation of state To compute general expressions for thermodynamic derivatives such as the specific heats and the adiabatic derivatives it is useful to write the equation of state in differential form, i.e.

$$\frac{\mathrm{d}P}{P} = \chi_T \frac{\mathrm{d}T}{T} + \chi_\rho \frac{\mathrm{d}\rho}{\rho},\tag{3.48}$$

where χ_T and χ_ρ are defined as

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho, X_i} = \frac{T}{P} \left(\frac{\partial P}{\partial T}\right)_{\rho, X_i},\tag{3.49}$$

$$\chi_{\rho} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{T,X_{i}} = \frac{\rho}{P} \left(\frac{\partial P}{\partial \rho}\right)_{T,X_{i}}.$$
(3.50)

The subscript X_i means that the composition is held constant as well. In a general equation of state χ_T and χ_ρ can depend on T and ρ themselves, but if they are (approximately) constant then we can write the equation of state in power-law form:

 $P = P_0 \rho^{\chi_\rho} T^{\chi_T}.$

For example, for an ideal gas without radiation we have $\chi_T = \chi_\rho = 1$, while for a radiation-dominated gas $\chi_T = 4$ and $\chi_\rho = 0$.

3.4.1 Specific heats

The specific heats at constant volume c_V and at constant pressure c_P for a unit mass of gas follow from eq. (3.47):

$$c_V = \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v,\tag{3.51}$$

$$c_P = \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_P,\tag{3.52}$$

where a partial derivative taken at constant v is the same as one taken at constant ρ . For an ideal gas, with $u = U/\rho = \frac{3}{2}P/\rho$, we obtain from eq. (3.21) the familiar result $c_V = \frac{3}{2}R/\mu$. For a radiationdominated gas, eq. (3.43) yields $c_V = 4aT^3/\rho$. Using thermodynamic transformations and some algebraic manipulation (see Appendix 3.A), it follows quite generally that the specific heats are related by

$$c_P - c_V = \frac{P}{\rho T} \frac{\chi_T^2}{\chi_\rho}.$$
(3.53)

For an ideal gas this amounts to $c_P - c_V = \mathcal{R}/\mu$, and therefore $c_P = \frac{5}{2}\mathcal{R}/\mu$. For a radiation-dominated gas $\chi_{\rho} = 0$ and hence $c_P \to \infty$: indeed, since P_{rad} only depends on T, a change in temperature cannot be performed at constant pressure.

The ratio of specific heats is denoted as γ :

$$\gamma = \frac{c_P}{c_V} = 1 + \frac{P}{\rho T c_V} \frac{\chi_T^2}{\chi_\rho},\tag{3.54}$$

so that $\gamma = \frac{5}{3}$ for an ideal gas.

Expressions for dq It is often useful to have expressions for the change in heat content dq (eq. 3.47) in terms of variations of T and ρ or T and P. Making use of the specific heats one can derive (see Appendix 3.A)

$$dq = T ds = c_V dT - \chi_T \frac{P}{\rho^2} d\rho = c_P dT - \frac{\chi_T}{\chi_\rho} \frac{dP}{\rho}.$$
(3.55)

3.4.2 Adiabatic derivatives

The thermodynamic response of a system to adiabatic changes is measured by the so-called *adiabatic derivatives*. Two of these have special importance for stellar structure:

• The *adiabatic exponent*² γ_{ad} measures the response of the pressure to adiabatic compression or expansion, i.e. to a change in the density. It is defined as

$$\gamma_{\rm ad} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\rm ad} \tag{3.56}$$

where the subscript 'ad' means that the change is performed adiabatically, that is, at constant entropy. If γ_{ad} is constant then $P \propto \rho^{\gamma_{ad}}$ for adiabatic changes. As we shall see later, γ_{ad} is related to the *dynamical stability* of stars.

• The adiabatic temperature gradient is defined as

$$\nabla_{\rm ad} = \left(\frac{\partial \log T}{\partial \log P}\right)_{\rm ad} \tag{3.57}$$

It is in fact another exponent that describes the behaviour of the temperature under adiabatic compression or expansion ($T \propto P^{\nabla_{ad}}$ if ∇_{ad} is constant), which turns out to be important for stability against *convection*.

The adiabatic exponent For an adiabatic process dq = 0 in eq. (3.47) and therefore

$$du = \frac{P}{\rho^2} d\rho.$$
(3.58)

We have seen in Sect. 3.3.1 that for a perfect gas of free particles the internal energy density U is proportional to P, in both the NR and ER limits. For such a simple system we can therefore write, as we did in Sect. 2.3,

$$u = \phi \frac{P}{\rho} \tag{3.59}$$

with ϕ a constant (between $\frac{3}{2}$ and 3). If we differentiate this and substitute into eq. (3.58) we obtain for an adiabatic change

$$\frac{\mathrm{d}P}{P} = \frac{\phi + 1}{\phi} \frac{\mathrm{d}\rho}{\rho}.$$
(3.60)

Therefore, according to the definition of γ_{ad} (eq. 3.56),

$$\gamma_{\rm ad} = \frac{\phi + 1}{\phi}$$
 (for a simple, perfect gas). (3.61)

$$\Gamma_{1} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\mathrm{ad}} = \gamma_{\mathrm{ad}}, \qquad \qquad \frac{\Gamma_{2}}{\Gamma_{2} - 1} = \left(\frac{\partial \log P}{\partial \log T}\right)_{\mathrm{ad}} = \frac{1}{\nabla_{\mathrm{ad}}}, \qquad \qquad \Gamma_{3} = \left(\frac{\partial \log T}{\partial \log \rho}\right)_{\mathrm{ad}} + 1.$$

They obey the relation

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

²In many textbooks one finds instead the adiabatic exponents Γ_1 , Γ_2 , and Γ_3 introduced by Chandrasekhar. They are defined, and related to γ_{ad} and ∇_{ad} , as follows:

- for *non-relativistic* particles (e.g. a classical ideal gas, NR degenerate electrons) $\phi = \frac{3}{2}$ and therefore $\gamma_{ad} = \frac{5}{3}$
- for *extremely relativistic* particles (e.g. photons, ER degenerate electrons) $\phi = 3$ and therefore $\gamma_{ad} = \frac{4}{3}$
- for a mixture of gas and radiation $(0 \le \beta \le 1)$ and/or moderately relativistic degenerate electrons, $\frac{4}{3} \le \gamma_{ad} \le \frac{5}{3}$

For a general equation of state, described by eq. (3.48), one can derive (see Appendix 3.A)

$$\gamma_{\rm ad} = \chi_{\rho} + \frac{P}{\rho T c_V} \chi_T^2. \tag{3.62}$$

Therefore γ_{ad} is related to the ratio of specific heats (eq. 3.54), $\gamma_{ad} = \gamma \chi_{\rho}$. The γ 's are equal if $\chi_{\rho} = 1$ (as in the case of an ideal gas).

The adiabatic temperature gradient By writing eq. (3.56) as $dP/P = \gamma_{ad} d\rho/\rho$ for an adiabatic change, and eliminating $d\rho$ with the help of eq. (3.48), we obtain a general relation between the adiabatic temperature gradient ∇_{ad} and the adiabatic exponent γ_{ad} :

$$\nabla_{\rm ad} = \frac{\gamma_{\rm ad} - \chi_{\rho}}{\gamma_{\rm ad} \chi_T},\tag{3.63}$$

This gives the following limiting cases:

- for an ideal gas without radiation ($\beta = 1$) we have $\chi_T = \chi_\rho = 1$, which together with $\gamma_{ad} = \frac{5}{3}$ gives $\nabla_{ad} = \frac{2}{5} = 0.4$.
- for a radiation-dominated gas $(\beta = 0) \chi_T = 4$ and $\chi_{\rho} = 0$ so that $\nabla_{ad} = \frac{1}{4} = 0.25$.

For a general equation of state one has to consider the general expression for γ_{ad} (eq. 3.62) in eq. (3.63). From the expression of dq in terms of dT and dP (3.55) it follows that

$$\nabla_{\rm ad} = \frac{P}{\rho T c_P} \frac{\chi_T}{\chi_\rho}.$$
(3.64)

This means that for a general non-adiabatic process we can write eq. (3.55) as

$$dq = c_P \left(dT - \nabla_{ad} \frac{T}{P} dP \right), \tag{3.65}$$

which will prove to be a useful relation later on.

We give some important results without derivations, which can be found in K&W Chapters 13.2 and 16.3 or in HANSEN Chapter 3.7:

- for a mixture of gas and radiation with $0 < \beta < 1$, ∇_{ad} and γ_{ad} both depend on β and take on intermediate values, i.e. $0.25 < \nabla_{ad} < 0.4$.
- for a non-relativistic degenerate gas, we have to consider that although electrons dominate the pressure, there is a (tiny) temperature dependence due to the ion gas which must be taken into account in calculating χ_T and therefore ∇_{ad} . After some manipulation it can be shown that in this case $\nabla_{ad} = 0.4$, as for the ideal classical gas.
- for an extremely relativistic degenerate gas one also has to consider that while the electrons are relativistic, the ions are still non-relativistic. It turns out that in this limit $\nabla_{ad} = 0.5$.

3.5 Ionization

We have so far implicitly assumed complete ionization of the gas, i.e. that it consists of bare atomic nuclei and free electrons. This is a good approximation in hot stellar interiors, where $T > 10^6$ K so that typical energies kT are much larger than the energy needed to ionize an atom, i.e. to knock off a bound electron. In the cooler outer layers of a star, however, we need to consider the *partial ionization* of the elements. In this case quasi-static changes of the state variables (ρ and T) will lead to changes in the degree of ionization. This can have a large effect on the thermodynamic properties of the gas, e.g. on γ_{ad} and ∇_{ad} .

In LTE the number densities of ionized and neutral species are determined by the Saha equation

$$\frac{n_{r+1}}{n_r}n_e = \frac{u_{r+1}}{u_r} \frac{2(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}$$
(3.66)

where n_r and n_{r+1} indicate the number densities of r and r+1 times ionized nuclei, χ_r is the ionization potential, i.e. the energy required to remove the r-th bound electron, and u_r and u_{r+1} are the partition functions. The partition functions depend on T but can in most cases be approximated by the statistical weights of the ground states of the bound species. (This equation can be derived from statistical mechanics, e.g. see K&W Chapter 14.1.)

3.5.1 Ionization of hydrogen

As an example, we consider the simple case where the gas consists only of hydrogen. Then there are just three types of particle, electrons and neutral and ionized hydrogen, with $u_{\rm H} = u_0 = 2$ and $u_{\rm H+} = u_1 = 1$. We write their number densities as n_+ and n_0 so that

$$\frac{n_{+}}{n_{0}}n_{e} = \frac{(2\pi m_{e}kT)^{3/2}}{h^{3}}e^{-\chi_{\rm H}/kT}$$
(3.67)

where $\chi_{\rm H} = 13.6 \,\text{eV}$. The gas pressure is given by $P_{\rm gas} = (n_0 + n_+ + n_{\rm e}) kT$ and the density is $\rho = (n_0 + n_+) m_u$. The degree of ionization is defined as

$$x = \frac{n_+}{n_0 + n_+} \tag{3.68}$$

so that P_{gas} can be written in terms of the degree of ionization

$$P_{\rm gas} = (1+x)\mathcal{R}\rho T \tag{3.69}$$

We can then rewrite Saha's equation as

$$\frac{x^2}{1-x^2} = \frac{(2\pi m_{\rm e})^{3/2}}{h^3} \frac{(kT)^{5/2}}{P_{\rm gas}} e^{-\chi_{\rm H}/kT}$$
(3.70)

We see that the degree of ionization increases with T, as expected since more atoms are broken up by the energetic photons. However, x decreases with gas pressure (or density) when T is kept constant, because this increases the probability of recombination which is proportional to n_e . From eq. (3.69) we see that the mean molecular weight $\mu = 1/(1 + x)$ decreases as hydrogen becomes ionized (one atomic mass is divided over two particles).

To estimate the effect on the thermodynamic properties of the gas, we note that in the case of partial ionization the internal energy has a contribution from the available potential energy of recombination. Per unit volume this contribution is equal to $n_+ \chi_{\rm H}$, so per unit mass it equals $n_+ \chi_{\rm H} / \rho = x \chi_{\rm H} / m_{\rm u}$. Thus

$$u = \frac{3}{2} \frac{P_{\text{gas}}}{\rho} + x \frac{\chi_{\text{H}}}{m_{\text{u}}} = \frac{3}{2} (1+x) \mathcal{R}T + x \frac{\chi_{\text{H}}}{m_{\text{u}}}.$$
(3.71)



Figure 3.5. The adiabatic temperature gradient ∇_{ad} plotted against temperature. The left panel shows the effect of partial ionization for the simple case of a pure hydrogen gas, for three values of the density $(10^{-4}, 10^{-6} \text{ and } 10^{-8} \text{ g/cm}^3)$. When hydrogen is partially ionized, ∇_{ad} is decreased below its ideal-gas value of 0.4. The circles indicate the points where the degree of ionization x = 0.5, close to the minimum of ∇_{ad} . As the density increases, a higher temperature is needed to reach the same ionization degree. The right panel shows how ∇_{ad} varies with temperature in a detailed stellar model of $1 M_{\odot}$, between the surface (at $T \approx 6000 \text{ K}$) and the centre (at $T \approx 1.5 \times 10^7 \text{ K}$). Apart from the hydrogen ionization zone around 10^4 K , a second depression of ∇_{ad} around 10^5 K is seen which is due to the first ⁴He ionization zone. The second He ionization zone is merged with H ionization because it occurs at similar temperatures and densities. Note that the region where $T < 10^6 \text{ K}$ comprises only the outer 1% of the mass of the Sun. (The dotted line shows how ∇_{ad} would vary with T in this model if the composition were pure hydrogen, as was assumed in the left panel.)

A small increase in temperature increases the degree of ionization, which results in a large amount of energy being absorbed by the gas. In other words, the *specific heat* of a partially ionized gas will be much larger than for an unionized gas, or for a completely ionized gas (in the latter case x = 1 so that the second term in eq. (3.71) becomes a constant and therefore irrelevant).

Now consider what happens if the gas is adiabatically compressed. Starting from neutral hydrogen, for which $\nabla_{ad} = 0.4$, the temperature initially increases as $T \propto P^{0.4}$. Further compression (work done on the gas) increases *u*, but when partial ionization sets in most of this energy goes into raising the degree of ionization (second term of eq. 3.71) and only little into raising the temperature (first term). In other words, *T* increases less strongly with with *P*, and therefore $\nabla_{ad} < 0.4$. A detailed calculation (e.g. see K&W Chapter 14.3) shows that under typical conditions ∇_{ad} reaches a minimum value of ≈ 0.1 when $x \approx 0.5$. As the gas becomes almost fully ionized, ∇_{ad} rises back to 0.4. The variation of ∇_{ad} with temperature for a pure hydrogen gas is shown in the left panel of Fig. 3.5 for different values of the density.

The decrease of ∇_{ad} in partial ionization zones can induce *convection* in the outer layers of stars, as we shall see in Ch. 5. Similarly it can be shown that γ_{ad} decreases in partial ionization zones, from $\frac{5}{3}$ to $\gamma_{ad} \approx 1.2$ when $x \approx 0.5$. This has consequences for the stability of stars, as we shall also see.



Figure 3.6. Schematic depiction of the electrostatic potential of an isolated ion (left) and the superposition of the potentials of neighbouring ions (right). Figure reproduced from KIPPENHANHN & WEIGERT.

3.5.2 Ionization of a mixture of gases

In a mixture of gases the situation becomes more complicated because many, partly ionized species have to be considered, the densities of which all depend on each other (see e.g. K&W Chapter 14.4-14.5). However the basic physics remains the same as considered above for the simple case of pure hydrogen. The effect on the thermodynamic properties is that e.g. ∇_{ad} can show additional deviations below 0.4 at different temperatures, especially where helium (the second-most abundant element in stars) is partially ionized. This is illustrated in Fig. 3.5b which shows the variation of ∇_{ad} with temperature in a homogeneous model for the initial Sun.

3.5.3 Pressure ionization

As ρ increases indefinitely, the Saha equation gives $x \to 0$, i.e. ionized gas recombines to form atoms. This is obviously nonsense at very high density, and becomes incorrect when the average distance *d* between ions becomes less than an atomic radius. In this situation the ionization energy is suppressed (there are fewer bound excited states; see Fig. 3.6), a situation known as *pressure ionization*.

Consider the case of hydrogen: the volume per H atom is $1/n_{\rm H}$ so that $d = (\frac{4\pi}{3}n_{\rm H})^{-1/3}$. Pressure ionization sets in when $d \leq a_0 = 5 \times 10^{-9}$ cm (the Bohr radius). This implies

$$n_{\rm H} \gtrsim \frac{1}{\frac{4\pi}{3}a_0^3}$$

or $\rho = n_{\rm H}m_{\rm H} \gtrsim 3 \,{\rm g \, cm^{-3}}$. Other elements are pressure-ionized at similar values of the density, within an order of magnitude. At densities $\gtrsim 10 \,{\rm g \, cm^{-3}}$, therefore, we can again assume complete ionization.

Fig. 3.7 shows the approximate boundary in the density-temperature diagram between neutral and ionized hydrogen according the Saha equation for $\rho < 1 \text{ g cm}^{-3}$, and as a result of pressure ionization at higher densities.

3.6 Other effects on the equation of state

3.6.1 Coulomb interactions and crystallization

We have so far ignored the effect of electrostatic or Coulomb interactions between the ions and electrons in the gas. Is this a reasonable approximation, i.e. are the interaction energies indeed small compared to the kinetic energies, as we have assumed in Sect. 3.3?

The average distance between gas particles (with mass Am_u) is $d \approx (\frac{4\pi}{3}n)^{-1/3}$ where *n* is the number density, $n = \rho/(Am_u)$. The typical Coulomb energy per particle (with charge Ze) is $\epsilon_C \approx Z^2 e^2/d$, while the average kinetic energy is $\epsilon_{kin} = \frac{3}{2}kT$. The ratio of Coulomb energy to kinetic energy is usually called the Coulomb parameter Γ_c , defined as

$$\Gamma_{\rm C} = \frac{Z^2 e^2}{d \, kT} = \frac{Z^2 e^2}{kT} \left(\frac{4\pi\rho}{3Am_{\rm u}}\right)^{1/3} = 2.275 \times 10^5 \, \frac{Z^2}{A^{1/3}} \, \frac{\rho^{1/3}}{T},\tag{3.72}$$



Figure 3.7. The equation of state in the ρ , T plane for a pure hydrogen gas. The dotted lines are the borders, also shown in Fig. 3.4, between regions where radiation, ideal gas and degenerate electrons dominate the pressure. The solid line shows where the ionization fraction of hydrogen is 0.5 according to the Saha equation, and where hydrogen becomes pressure-ionized at high density. The dashed lines show where the Coulomb interaction parameter $\Gamma_{\rm C}$ equals 1, above which Coulomb interactions become important, and where $\Gamma_{\rm C}$ = 170, above which the ions form a crystralline lattice. Above the dash-dotted line e⁺e⁻ pairs play an important role in stellar interiors.

where in the last equality the numerical factor is in cgs units. We see that Coulomb interactions increase in importance at high densities or low temperatures. Roughly, Coulomb interactions start to become important in stellar interiors when $\Gamma_C \gtrsim 1$.

To estimate the typical value of $\Gamma_{\rm C}$ in stellar interiors we approximate $\rho \approx \bar{\rho} = M/(\frac{4\pi}{3}R^3)$, and we approximate T by the average temperature estimated from the virial theorem, $T \approx \bar{T} \approx \frac{1}{3} \frac{Am_{\rm u}}{k} \frac{GM}{R}$ (eq. 2.29). Ignoring factors of order unity, we get

$$\Gamma_{\rm C} \approx 0.01 \frac{Z^2}{A^{4/3}} \left(\frac{M}{M_{\odot}}\right)^{-2/3}.$$
 (3.73)

The ratio $Z^2/A^{4/3}$ depends on the composition, and represents an average over the constituents of the gas. In stars mostly composed of hydrogen, $A \approx 1$ and $Z \approx 1$, and we find that in the Sun the Coulomb energy contributes of the order of 1% to the particle energies (and hence has a similar effect on the pressure). We are therefore justified in ignoring Coulomb interactions in stars similar to or more massive than the Sun. However, eq. (3.73) shows that in low-mass stars Coulomb interactions can start to contribute significantly. This can also be seen by comparing Fig. 3.4 and Fig. 3.7, where the location of the condition $\Gamma_{\rm C} = 1$ is indicated in the ρ -T diagram. Detailed models of low-mass stars need to take this effect into account. For $M \leq 10^{-3} M_{\odot}$ the Coulomb energies dominate. Such objects are not stars but planets (Jupiter's mass is about $10^{-3} M_{\odot}$). Calculations of the structure of planets requires a much more complicated equation of state than for stars.

Crystallization

If $\Gamma_C \gg 1$ the thermal motions of the ions are overwhelmed by the Coulomb interactions. In this situation the ions will tend to settle down into a conglomerate with a lower energy, in other words they will form a crystalline lattice. Detailed estimates indicate that this transition takes place at a critical value of $\Gamma_C \approx 170$. This condition is also indicated in Fig. 3.4 for a pure hydrogen gas. In reality, this situation will never occur in hydrogen-rich stellar interiors, but it can take place in cooling

white dwarfs (in which the temperature gradually decreases with time while the density remains constant). White dwarfs are usually composed of carbon and oxygen, so in this case we have to take into account the composition which raises the temperature at which the transition occurs (the 'melting' temperature) by a factor $Z^2/A^{1/3}$ according to eq. (3.72).

Finally we note that crystallization only occurs in the region where the electrons are strongly degenerate. You may verify that the Coulomb interaction energy between electrons and ions (Ze^2/d) is always smaller than the typical electron energy $(p_F^2/2m_e)$. The electrons therefore behave as a free degenerate gas, even if the ions form a crystalline structure.

3.6.2 Pair production

A very different process can take place at very high temperatures and relatively low densities. A photon may turn into an electron-positron pair if its energy hv exceeds the rest-mass energy of the pair, $hv > 2m_ec^2$. This must take place during the interaction with a nucleus, since otherwise momentum and energy cannot both be conserved. Pair production takes place at a typical temperature $kT \approx hv \approx 2m_ec^2$, or $T \approx 1.2 \times 10^{10}$ K. However, even at $T \sim 10^9$ K the number of energetic photons in the tail of the Planck distribution (eq. 3.41) is large enough to produce a large number of e^+e^- pairs. The newly created positrons tend to be annihilated quickly by the inverse reaction $(e^+ + e^- \rightarrow 2\gamma)$, as a result of which the number of positrons reaches equilibrium. At a few times 10^9 K, depending on the electron density, the number of positrons is a significant fraction of the number of electrons.

Pair production is similar to an ionization process: an increase in temperature leads to an increase in the number of particles at the expense of the photon energy (and pressure). Therefore pair production gives rise to a decrease of the adiabatic gradient γ_{ad} and of ∇_{ad} , similar to partial ionization. This is the main importance of pair production for stellar evolution: it affects the stability of very massive stars in advanced stages of evolution (when their temperature may reach values in excess of 10^9 K) and can trigger their collapse.

Suggestions for further reading

The contents of this chapter are also covered by Chapter 7 of MAEDER and by Chapters 13 to 16 of KIPPENHAHN & WEIGERT. However, a more elegant derivation of the equation of state, which is also more consistent with the way it is derived in these lecture notes, is given in Chapter 3 of HANSEN, KAWALER & TRIMBLE. Explicit expressions for many of the results that are only mentioned here can be found in this book.

Exercises

3.1 Conceptual questions

These questions are intended to test your understanding of the lectures. Try to answer them without referring to the lecture notes.

- (a) What do we mean by *local thermodynamic equilibrium* (LTE)? Why is this a good assumption for stellar interiors? What is the difference between LTE and *thermal equilibrium* (as treated in Ch. 2)?
- (b) In what type of stars does degeneracy become important? Is it important in main-sequence stars? Is it more important in high mass or low mass MS stars?

- (c) Explain qualitatively why for degenerate matter, the pressure increases with the density.
- (d) Why do electrons become relativistic when they are compressed into a smaller volume? Why does the pressure increase less steeply with the density in this case?
- (e) In the central region of a star we find free electrons and ions. Why do the electrons become degenerate first? Why do the ions never become degenerate in practice?

3.2 Mean molecular weight

Derive a general expression for the mean molecular weight of an ionized gas, as a function of composition X, Y, Z. Assume that, for elements heavier than H, nuclei are composed of equal number of protons and neutrons, so that the nuclear charge Z_i is half of the mass number A_i .

3.3 The ρ – *T* plane

Consider a gas of ionized hydrogen. In the ρ -T plane compute the approximate boundary lines between the regions where:

- (a) radiation pressure dominates,
- (b) the electrons behave like a classical ideal gas,
- (c) the electrons behave like a degenerate gas,
- (d) the electrons are relativistically degenerate.

3.4 The pressure of a gas of free particles

In this exercise you will derive some important relations from this chapter for yourself.

- (a) Suppose that the particles in a gas have momenta distributed as n(p) dp. Show that the pressure can be expressed by eq. (3.4).
- (b) For classical particles in LTE, the momentum distribution is given by the Maxwell-Boltzmann distribution, eq. (3.13). Calculate the pressure using eq. (3.4). Does the result look familiar?
- (c) Show that for a gas of free, non-relativistic particles $P = \frac{2}{3}U$ (eq. 3.11), where U is the internal energy density. Show that in the extremely relativistic limit $P = \frac{1}{3}U$ (eq. 3.12).
- (d) Electrons are fermions with 2 spin states. Explain why the maximum number of electrons per volume with momentum p can be written as eq. (3.28).
- (e) In the extreme case of complete degeneracy, $T \rightarrow 0$, the electrons fill up all available quantum states up to a maximum p_F , the Fermi momentum. Show that

$$p_{\rm F} = h \left(\frac{3n_{\rm e}}{8\pi}\right)^{\frac{1}{3}}$$

(f) Show that the pressure as function of the density for a non-relativistic degenerate electron gas can be written as

$$P = K_{\rm NR} \left(\frac{\rho}{\mu_{\rm e}}\right)^{x}$$

and derive an expression for $K_{\rm NR}$ and x.

(g) Show that the pressure as function of the density for an extremely relativistic degenerate electron gas can be written as

$$P = K_{\rm ER} \left(\frac{\rho}{\mu_{\rm e}}\right)^{\rm y}$$

and derive an expression for K_{ER} and y.

(h) Photons are bosons, and the distribution of their momenta is given by the Planck function (eq. 3.27). Show that in this case

 $U \propto T^4$

(Hint: to derive an expression for the proportionality constant a, you might want to use Mathematica or a list of standard integrals.)

(i) Now use (c) to show that the radiation pressure is given by $P_{\text{rad}} = \frac{1}{3}aT^4$.

3.5 Adiabatic derivatives

(a) Use the first law of thermodynamics to show that, for an ideal gas in an adiabatic process,

 $P \propto \rho^{\gamma_{\rm ad}}$ (3.74)

and give a value for the adiabatic exponent γ_{ad} .

(b) Use the ideal gas law in combination with eq. (3.74) to show that

$$\nabla_{\rm ad} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad,id}} = 0.4.$$

- (c) The quantity ∇_{ad} is referred to as the *adiabatic temperature gradient*. Normally you would use the term 'gradient of a quantity A' for dA/dr, or if you use mass coordinates instead of radius coordinates, dX/dm. Do you understand why ∇_{ad} can be referred to as a temperature 'gradient'?
- (d) (*) Show that for a mixture of an ideal gas plus radiation, the adiabatic exponent is given by

$$\gamma_{\rm ad} = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta},$$

where $\beta = P_{\text{gas}}/P$.

(Hints: write down the equation of state for the mixture in differential form as in eq. (3.48), and express χ_T and χ_ρ in terms of β . Then apply the first law of therrmodynamics for an adiabatic process.)

(e) (*) What is the value of γ_{ad} in the limit where radiation dominates and where pressure dominates? Does this look familiar?

3.6 Ionization effects

- (a) The particles in an ionized gas are charged and therefore undergo electrostatic (Coulomb) interactions. Why can can we nevertheless make the ideal-gas assumption in most stars (i.e. that the internal energy of the gas is just the sum of the kinetic energies of the particles)? For which stars do Coulomb interactions have a significant effect?
- (b) Why does the gas in the interior of a star become pressure-ionized at high densities?
- (c) Explain qualitatively why partial ionization leads to $\nabla_{ad} < \nabla_{ad,ideal} = 0.4$, in other words: why does adiabatic compression lead to a smaller temperature increase when the gas is partly ionized, compared to a completely ionized (or unionized) gas?

3.A Appendix: Themodynamic relations

In this Appendix we derive some of the thermodynamic relations that were given without proof in Chapter 3.

The first law of thermodynamics states that the heat added to a mass element of gas is the sum of the change in its internal energy and the work done by the mass element. Taking the element to be of unit mass, we can wite this as

$$dq = du + P dv = du - \frac{P}{\rho^2} d\rho, \qquad (3.75)$$

because the volume of a unit mass is $v = 1/\rho$. We can write the change in the internal energy of a unit mass in terms of the changes in the state variables (*T* and ρ) as

$$du = \left(\frac{\partial u}{\partial T}\right)_{\rho} dT + \left(\frac{\partial u}{\partial \rho}\right)_{T} d\rho.$$
(3.76)

The change in the entropy per unit mass, ds = dq/T, is therefore

$$ds = \frac{dq}{T} = \frac{1}{T} \left(\frac{\partial u}{\partial T} \right)_{\rho} dT + \frac{1}{T} \left[\left(\frac{\partial u}{\partial \rho} \right)_{T} - \frac{P}{\rho^{2}} \right] d\rho.$$
(3.77)

Because s is a function of state, $\partial^2 s / \partial \rho \partial T = \partial^2 s / \partial T \partial \rho$, which means that

$$\frac{1}{T}\frac{\partial^2 u}{\partial T \partial \rho} = \frac{\partial}{\partial T} \left[\frac{1}{T} \left(\frac{\partial u}{\partial \rho} \right)_T - \frac{P}{\rho^2 T} \right],\tag{3.78}$$

where the $\partial/\partial T$ on the right-hand side should be taken at constant ρ . Working out the right-hand side allows us to eliminate the second derivative of u, giving

$$\frac{1}{T^2} \left(\frac{\partial u}{\partial \rho} \right)_T = \frac{P}{\rho^2 T^2} - \frac{1}{\rho^2 T} \left(\frac{\partial P}{\partial T} \right)_\rho.$$

With the definition of χ_T (eq. 3.49) we can write $(\partial P/\partial T)_{\rho} = \chi_T P/T$, and thus

$$\left(\frac{\partial u}{\partial \rho}\right)_T = (1 - \chi_T) \frac{P}{\rho^2}.$$
(3.79)

Specific heats

The definitions of the specifc heats at constant volume and at constant pressure are

$$c_V \equiv \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_{\nu} = \left(\frac{\partial u}{\partial T}\right)_{\rho},\tag{3.80}$$

$$c_P \equiv \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_P. \tag{3.81}$$

To work out an expression for c_P , we need $(\partial u/\partial T)_P$ and $(\partial \rho/\partial T)_P$. To start with the latter, we use the differential form of the equation of state (3.48). At constant pressure dP = 0 this gives

$$\chi_{\rho} \frac{\mathrm{d}\rho}{\rho} = -\chi_T \frac{\mathrm{d}T}{T} \qquad \Rightarrow \qquad \left(\frac{\partial\rho}{\partial T}\right)_{\rho} = -\frac{\rho}{T} \frac{\chi_T}{\chi_{\rho}}.$$
 (3.82)

To obtain an expression for $(\partial u/\partial T)_P$ we use eq. (3.76), which we can write as

$$\frac{\mathrm{d}u}{\mathrm{d}T} = \left(\frac{\partial u}{\partial T}\right)_{\rho} + \left(\frac{\partial u}{\partial \rho}\right)_{T} \frac{\mathrm{d}\rho}{\mathrm{d}T}$$

and therefore

$$\left(\frac{\partial u}{\partial T}\right)_{P} = \left(\frac{\partial u}{\partial T}\right)_{\rho} + \left(\frac{\partial u}{\partial \rho}\right)_{T} \left(\frac{\partial \rho}{\partial T}\right)_{P} = \left(\frac{\partial u}{\partial T}\right)_{\rho} + (\chi_{T} - 1)\frac{\chi_{T}}{\chi_{\rho}}\frac{P}{\rho T}.$$
(3.83)

To obtain the last equality we used eqs. (3.79) and (3.82). From the definitions (3.80) and (3.81) we thus arrive at the following relation between c_P and c_V :

$$c_P - c_V = \frac{\chi_T^2}{\chi_\rho} \frac{P}{\rho T}$$
(3.84)

which is eq. (3.53).

Expressions for dq

It is useful to be able to write the change in heat content of a unit mass in terms of the changes in the state variables. Eq. (3.77) already shows how dq is written in terms of T and ρ , i.e.

$$dq = T ds = c_V dT - \chi_T \frac{P}{\rho^2} d\rho, \qquad (3.85)$$

making use of (3.79) and (3.80). It is often useful to express dq is terms of T and P, rather than ρ . To do this we write $d\rho$ with the help of eq. (3.48),

$$d\rho = \frac{\rho}{\chi_{\rho}} \left(\frac{dP}{P} - \chi_T \frac{dT}{T} \right), \tag{3.86}$$

so that

$$dq = c_V dT - \frac{\chi_T}{\chi_\rho} \frac{P}{\rho} \left(\frac{dP}{P} - \chi_T \frac{dT}{T} \right) = \left(c_V + \frac{\chi_T^2}{\chi_\rho} \frac{P}{\rho T} \right) dT - \frac{\chi_T}{\chi_\rho} \frac{1}{\rho} dP.$$
(3.87)

The terms with parentheses in the last equality are simple c_P , according to (3.84), and therefore

$$dq = T ds = c_P dT - \frac{\chi_T}{\chi_\rho} \frac{dP}{\rho}.$$
(3.88)

Adiabatic derivatives

Eq. (3.88) makes it easy to derive an expression for the adiabatic temperature gradient (3.57),

$$\nabla_{\rm ad} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\rm ad}.\tag{3.89}$$

An adiabatic change in T and P means the changes take place at constant s, or with dq = 0. Hence (3.88) shows that

$$\left(\frac{\mathrm{d}T}{\mathrm{d}P}\right)_{s} = \frac{1}{\rho c_{P}} \frac{\chi_{T}}{\chi_{\rho}} \qquad \Rightarrow \qquad \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{s} = \frac{P}{T} \left(\frac{\mathrm{d}T}{\mathrm{d}P}\right)_{s} = \frac{P}{\rho T c_{P}} \frac{\chi_{T}}{\chi_{\rho}}.$$
(3.90)

This means

$$\nabla_{\rm ad} = \frac{P}{\rho T c_P} \frac{\chi_T}{\chi_\rho} \,, \tag{3.91}$$

which is eq. (3.64). With the help of this expression we can also write (3.88) as

$$dq = c_P \left(dT - \nabla_{ad} \frac{T}{P} dP \right).$$
(3.92)

To derive an expression for the adiabatic exponent (3.56),

$$\gamma_{\rm ad} \equiv \left(\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\rho}\right)_{\rm ad},\tag{3.93}$$

we use (3.85) and (3.88) and set dq = 0 in both expressions. This gives

$$dT = \frac{P}{\rho^2 c_V} \chi_T d\rho$$
 and $dT = \frac{1}{\rho c_P} \frac{\chi_T}{\chi_\rho} dP$.

Eliminating dT from both expressions gives

$$\frac{\mathrm{d}P}{P} = \frac{c_P}{c_V} \chi_\rho \frac{\mathrm{d}\rho}{\rho} \qquad \Rightarrow \qquad \left(\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\rho}\right)_s = \frac{c_P}{c_V} \chi_\rho.$$

This means

$$\gamma_{\rm ad} = \frac{c_P}{c_V} \chi_\rho = \gamma \chi_\rho \quad , \tag{3.94}$$

where $\gamma = c_P/c_V$ is the ratio of specific heats. Using eq. (3.84) this can also be written as

$$\gamma_{\rm ad} = \chi_{\rho} + \frac{P}{\rho T c_V} \chi_T^2, \tag{3.95}$$

which is eq. (3.62).

Chapter 4

Polytropic stellar models

As mentioned in Sec. 2.2, the equation of hydrostatic equilibrium can be solved if the pressure is a known function of the density, $P = P(\rho)$. In this situation the mechanical structure of the star is completely determined. A special case of such a relation between P and ρ is the *polytropic relation*,

$$P = K\rho^{\gamma} \tag{4.1}$$

where K and γ are both constants. The resulting stellar models are known as *polytropic stellar models* or simply polytropes. Polytropic models have played an important role in the historical development of stellar structure theory. Although nowadays their practical use has mostly been superseded by more realistic stellar models, due to their simplicity polytropic models still give useful insight into several important properties of stars. Moreover, in some cases the polytropic relation is a good approximation to the real equation of state. We have encountered a few examples of polytropic equations of state in Chapter 3, e.g. the pressure of degenerate electrons, and the case where pressure and density are related adiabatically.

In this brief chapter – and the accompanying computer practicum – we will derive the analytic theory of polytropes and construct polytropic models, and study to which kind of stars they correspond, at least approximately.

4.1 Polytropes and the Lane-Emden equation

If the equation of state can be written in polytropic form, the equations for mass continuity (dm/dr, eq. 2.3) and for hydrostatic equilibrium (dP/dr, eq. 2.12) can be combined with eq. (4.1) to give a second-order differential equation for the density:

$$\frac{1}{\rho r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \rho^{\gamma - 2} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right) = -\frac{4\pi G}{K\gamma} \tag{4.2}$$

The exponent γ is often replaced by the so-called polytropic index *n*, which is defined by

$$n = \frac{1}{\gamma - 1} \qquad \text{or} \qquad \gamma = 1 + \frac{1}{n} \tag{4.3}$$

In order to construct a polytropic stellar model we have to solve eq. (4.2), together with two boundary conditions which are set in the centre, r = 0:

$$\rho(0) = \rho_c \quad \text{and} \quad \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{r=0} = 0,$$
(4.4)

where ρ_c is a parameter to be chosen, or determined from other constraints.

n	z_n	Θ_n	$ ho_{ m c}/ar ho$	N_n	W_n
0	2.44949	4.89898	1.00000		0.119366
1	3.14159	3.14159	3.28987	0.63662	0.392699
1.5	3.65375	2.71406	5.99071	0.42422	0.770140
2	4.35287	2.41105	11.40254	0.36475	1.638183
3	6.89685	2.01824	54.1825	0.36394	11.05068
4	14.97155	1.79723	622.408	0.47720	247.559
4.5	31.8365	1.73780	6189.47	0.65798	4921.84
5	00	1.73205	~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~	∞	00

Table 4.1. Numerical values for polytropic models with index *n*.

In order to simplify eq. (4.2), we define two new dimensionless variables w (related to the density) and z (related to the radius) by writing

$$\rho = \rho_c w^n, \tag{4.5}$$

$$r = \alpha z$$
, with $\alpha = \left(\frac{n+1}{4\pi G} K \rho_c^{1/n-1}\right)^{1/2}$. (4.6)

This choice of α ensures that the constants K and $4\pi G$ are eliminated after substituting r and ρ into eq. (4.2). The resulting second-order differential equation is called the *Lane-Emden equation*:

$$\frac{1}{z^2} \frac{\mathrm{d}}{\mathrm{d}z} \left(z^2 \frac{\mathrm{d}w}{\mathrm{d}z} \right) + w^n = 0. \tag{4.7}$$

A polytropic stellar model can be constructed by integrating this equation outwards from the centre. The boundary conditions (4.4) imply that in the centre (z = 0) we have w = 1 and dw/dz = 0. For n < 5 the solution w(z) is found to decrease monotonically and to reach zero at finite $z = z_n$, which corresponds to the surface of the model.

No general analytical solution of the Lane-Emden equation exists. The only exceptions are n = 0, 1 and 5, for which the solutions are:

$$n = 0:$$
 $w(z) = 1 - \frac{z^2}{6}$ $z_0 = \sqrt{6},$ (4.8)

$$n = 1:$$
 $w(z) = \frac{\sin z}{z}$ $z_1 = \pi,$ (4.9)

$$n = 5:$$
 $w(z) = \left(1 + \frac{z^2}{3}\right)^{-1/2}$ $z_5 = \infty.$ (4.10)

The case n = 0 ($\gamma = \infty$) corresponds to a homogeneous gas sphere with constant density ρ_c , following eq. (4.5). The solution for n = 5 is peculiar in that it has infinite radius; this is the case for all $n \ge 5$, while for $n < 5 z_n$ grows monotonically with n. For values of n other than 0, 1 or 5 the solution must be found by numerical integration (this is quite straightforward, see the accompanying computer practicum). Table 4.1 lists the value of z_n for different values of n, as well as several other properties of the solution that will be discussed below.

4.1.1 Physical properties of the solutions

Once the solution w(z) of the Lane-Emden equation is found, eq. (4.5) fixes the relative density distribution of the model, which is thus uniquely determined by the polytropic index n. Given the solution for a certain n, the physical properties of a polytropic stellar model, such as its mass and radius, are then determined by the parameters K and ρ_c , as follows.

The radius of a polytropic model follows from eq. (4.6):

$$R = \alpha z_n = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} z_n.$$
(4.11)

The mass m(z) interior to z can be obtained from integrating eq. (2.3), using eqs. (4.5), (4.6) and (4.7):

$$m(z) = \int_0^{\alpha z} 4\pi r^2 \rho \, dr = -4\pi \alpha^3 \rho_c \, z^2 \frac{\mathrm{d}w}{\mathrm{d}z}.$$
(4.12)

Hence the total mass of a polytropic model is

$$M = 4\pi\alpha^{3}\rho_{c}\,\Theta_{n} = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_{c}^{(3-n)/2n}\,\Theta_{n},\tag{4.13}$$

where we have defined Θ_n as

$$\Theta_n \equiv \left(-z^2 \frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n}.\tag{4.14}$$

By eliminating ρ_c from eqs. (4.11) and (4.13) we can find a relation between M, R and K,

$$K = N_n G M^{(n-1)/n} R^{(3-n)/n} \quad \text{with} \quad N_n = \frac{(4\pi)^{1/n}}{n+1} \Theta_n^{(1-n)/n} z_n^{(n-3)/n}.$$
(4.15)

Numerical values of Θ_n and N_n are given in Table 4.1. From the expressions above we see that n = 1 and n = 3 are special cases. For n = 1 the radius is independent of the mass, and is uniquely determined by the value of K. Conversely, for n = 3 the mass is independent of the radius and is uniquely determined by K. For a given K there is only one value of M for which hydrostatic equilibrium can be satisfied if n = 3.

The average density $\bar{\rho} = M/(\frac{4}{3}\pi R^3)$ of a polytropic star is related to the central density by eqs. (4.11) and (4.13) as

$$\bar{\rho} = \left(-\frac{3}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n}\rho_c = \frac{3\Theta_n}{z_n^3}\rho_c \tag{4.16}$$

Hence the ratio $\rho_c/\bar{\rho}$, i.e. the degree of central concentration of a polytrope, only depends on the polytropic index *n*. This dependence is also tabulated in Table 4.1. One may invert this relation to find the central density of a polytropic star of a given mass and radius.

The central pressure of a polytropic star follows from eq. (4.1), which can be written as

$$P_c = K \rho_c^{(n+1)/n}$$

In combination with (4.15) and (4.16) this gives

$$P_c = W_n \frac{GM^2}{R^4}$$
 with $W_n = \frac{z_n^4}{4\pi (n+1)\Theta_n^2}$. (4.17)

Note that in our simple scaling estimate, eq. (2.14), we found the same proportionality $P_c \propto GM^2/R^4$, where the proportionality constant W_n is now determined by the polytropic index *n* (see Table 4.1). We can eliminate *R* in favour of ρ_c to obtain the very useful relation

$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$
 with $C_n = \frac{(4\pi)^{1/3}}{n+1} \Theta_n^{-2/3}$, (4.18)

where you may verify that the constant C_n is only weakly dependent on *n*, unlike W_n in (4.17).

We give without derivation an expression for the gravitational potential energy of a polytrope of index *n*:

$$E_{\rm gr} = -\frac{3}{5-n} \frac{GM^2}{R}.$$
 (4.19)

(The derivation can be found in K&W Sec. 19.9 and MAEDER Sec. 24.5.1.)

4.2 Application to stars

Eq. (4.15) expresses a relation between the constant K in eq. (4.1) and the mass and radius of a polytropic model. This relation can be interpreted in two very different ways:

- The constant K may be given in terms of physical constants. This is the case, for example, for a star dominated by the pressure of degenerate electrons, in either the non-relativistic limit or the extremely relativistic limit. In that case eq. (4.15) defines a unique relation between the mass and radius of a star.
- In other cases the constant K merely expresses proportionality in eq. (4.1), i.e. K is a free parameter that is constant in a particular star, but may vary from star to star. In this case there are many different possible values of M and R. For a star with a given mass and radius, the corresponding value of K for this star can be determined from eq. (4.15).

In this section we briefly discuss examples for each of these two interpretations.

4.2.1 White dwarfs and the Chandrasekhar mass

Stars that are so compact and dense that their interior pressure is dominated by degenerate electrons are known observationally as *white dwarfs*. They are the remnants of stellar cores in which hydrogen has been completely converted into helium and, in most cases, also helium has been fused into carbon and oxygen. Since the pressure of a completely degenerate electron gas is a function of density only (Sec. 3.3.5), the mechanical structure if a white dwarf is fixed and is independent of temperature. We can thus understand some of the structural properties of white dwarfs by means of polytropic models.

We start by considering the equation of state for a degenerate, non-relativistic electron gas. From eq. (3.35) this can be described by a polytropic relation with n = 1.5. Since the corresponding K is determined by physical constants, eq. (4.15) shows that such a polytrope follows a mass-radius relation of the from

$$R \propto M^{-1/3}.\tag{4.20}$$

More massive white dwarfs are thus more compact, and therefore have a higher density. Above a certain density the electrons will become relativistic as they are pushed up to higher momenta by the Pauli exclusion principle. The degree of relativity increases with density, and therefore with the mass of the white dwarf, until at a certain mass all the electrons become extremely relativistic, i.e., their

speed $v_e \rightarrow c$. In this limit the equation of state has changed from eq. (3.35) to eq. (3.37), which is also a polytropic relation but with n = 3. We have already seen above that an n = 3 polytrope is special in the sense that it has a unique mass, which is determined by K and is independent of the radius:

$$M = 4\pi \Theta_3 \left(\frac{K}{\pi G}\right)^{3/2}.$$
(4.21)

This value corresponds to an upper limit to the mass of a gas sphere in hydrostatic equilibrium that can be supported by degenerate electrons, and thus to the maximum possible mass for a white dwarf. Its existence was first found by Chandrasekhar in 1931, after whom this limiting mass was named. Substituting the proper numerical values into eq. (4.21), with *K* corresponding to eq. (3.37), we obtain the *Chandrasekhar mass*

 $M_{\rm Ch} = 5.836 \,\mu_e^{-2} \,M_{\odot}. \tag{4.22}$

White dwarfs are typically formed of helium, carbon or oxygen, for which $\mu_e = 2$ and therefore $M_{\rm Ch} = 1.46 M_{\odot}$. Indeed no white dwarf with a mass exceeding this limit is known to exist.

4.2.2 Eddington's standard model

As an example of a situation where K is not fixed by physical constants but is essentially a free parameter, we consider a star in which the pressure is given by a mixture of ideal gas pressure and radiation pressure, eq. (3.45). In particular we make the assumption that the ratio β of gas pressure to total pressure is constant, i.e. has the same value in each layer of the star. Since $P_{\text{gas}} = \beta P$ we can write

$$P = \frac{1}{\beta} \frac{\mathcal{R}}{\mu} \rho T, \tag{4.23}$$

while also

Ì

$$1 - \beta = \frac{P_{\rm rad}}{P} = \frac{aT^4}{3P}.$$
 (4.24)

Thus the assumption of constant β means that $T^4 \propto P$ throughout the star. If we substitute the complete expression for T^4 into eq. (4.24) we obtain

$$P = \left(\frac{3\mathcal{R}^4}{a\mu^4} \frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3},$$
(4.25)

which is a polytropic relation with n = 3 for constant β . Since we are free to choose β between 0 and 1, the constant K is indeed a free parameter dependent on β .

The relation (4.25) was derived by Arthur Eddington in the 1920s for his famous 'standard model'. He found that in regions with a high opacity κ (see Ch. 5) the ratio of local luminosity to mass coordinate l/m is usually small, and vice versa. Making the assumption that $\kappa l/m$ is constant throughout the star is equivalent to assuming that β is constant (again, see Ch. 5). Indeed, for stars in which radiation is the main energy transport mechanism this turns out to be approximately true, even though it is a very rough approximation to the real situation. Nevertheless, the structure of stars on the main sequence with $M \gtrsim M_{\odot}$ is reasonably well approximated by that of a n = 3 polytrope. Since the mass of a n = 3 polytrope is given by eq. (4.21), we see from eq. (4.25) that there is a unique relation between the mass M of a star and β . The relative contribution of radiation pressure increases with the mass of a star. This was also noted by Eddington, who pointed out that the limited range of known stellar masses corresponds to values of β that are significantly different from 0 or 1.

Suggestions for further reading

Polytropic stellar models are briefly covered in Chapter 24.5 of MAEDER and treated more extensively in Chapter 19 of KIPPENHAHN & WEIGERT and Chapter 7.2 of HANSEN.

Exercises

4.1 The Lane-Emden equation

- (a) Derive eq. (4.2) from the stellar structure equations for mass continuity and hydrostatic equilibrium. (Hint: multiply the hydrostatic equation by r^2/ρ and take the derivative with respect to r).
- (b) What determines the second boundary condition of eq. (4.4), i.e., why does the density gradient have to vanish at the center?
- (c) By making the substitutions (4.3), (4.5) and (4.6), derive the Lane-Emden equation (4.7).
- (d) Solve the Lane-Emden equation analytically for the cases n = 0 and n = 1.

4.2 Polytropic models

- (a) Derive K and γ for the equation of state of an ideal gas at a fixed temperature T, of a non-relativistic degenerate gas and of a relativistic degenerate gas.
- (b) Using the Lane-Emden equation, show that the mass distribution in a polytropic star is given by eq. (4.12), and show that this yields eq. (4.13) for the total mass of a polytrope.
- (c) Derive the expressions for the central density ρ_c and the central pressure P_c as function of mass and radius, eqs. (4.16) and (4.17).
- (d) Derive eq. (4.18) and compute the constant C_n for several values of n.

4.3 White dwarfs

To understand some of the properties of white dwarfs (WDs) we start by considering the equation of state for a degenerate, non-relativistic electron gas.

- (a) What is the value of K for such a star? Remember to consider an appropriate value of the mean molecular weight per free electron μ_e .
- (b) Derive how the central density ρ_c depends on the mass of a non-relativistic WD. Using this with the result of Exercise 4.2(b), derive a radius-mass relation R = R(M). Interpret this physically.
- (c) Use the result of (b) to estimate for which WD masses the relativistic effects would become important.
- (d) Show that the derivation of a R = R(M) relation for the extreme relativistic case leads to a unique mass, the so-called *Chandrasekhar mass*. Calculate its value, i.e. derive eq. (4.22).

4.4 Eddington's standard model

(a) Show that for constant β the virial theorem leads to

$$E_{\rm tot} = \frac{\beta}{2} E_{\rm gr} = -\frac{\beta}{2-\beta} E_{\rm int},\tag{4.26}$$

for a classical, non-relativistic gas. What happens in the limits $\beta \to 1$ and $\beta \to 0$?

(b) Verify eq. (4.25), and show that the corresponding constant K depends on β and the mean molecular weight μ as

$$K = \frac{2.67 \times 10^{15}}{\mu^{4/3}} \left(\frac{1-\beta}{\beta^4}\right)^{1/3}.$$
(4.27)

(c) Use the results from above and the fact that the mass of an n = 3 polytrope is uniquely determined by K, to derive the relation $M = M(\beta, \mu)$. This is useful for numerically solving the amount of radiation pressure for a star with a given mass.