Basic Physics and Radiative Processes

Ricardo Chávez Murillo

April 2021

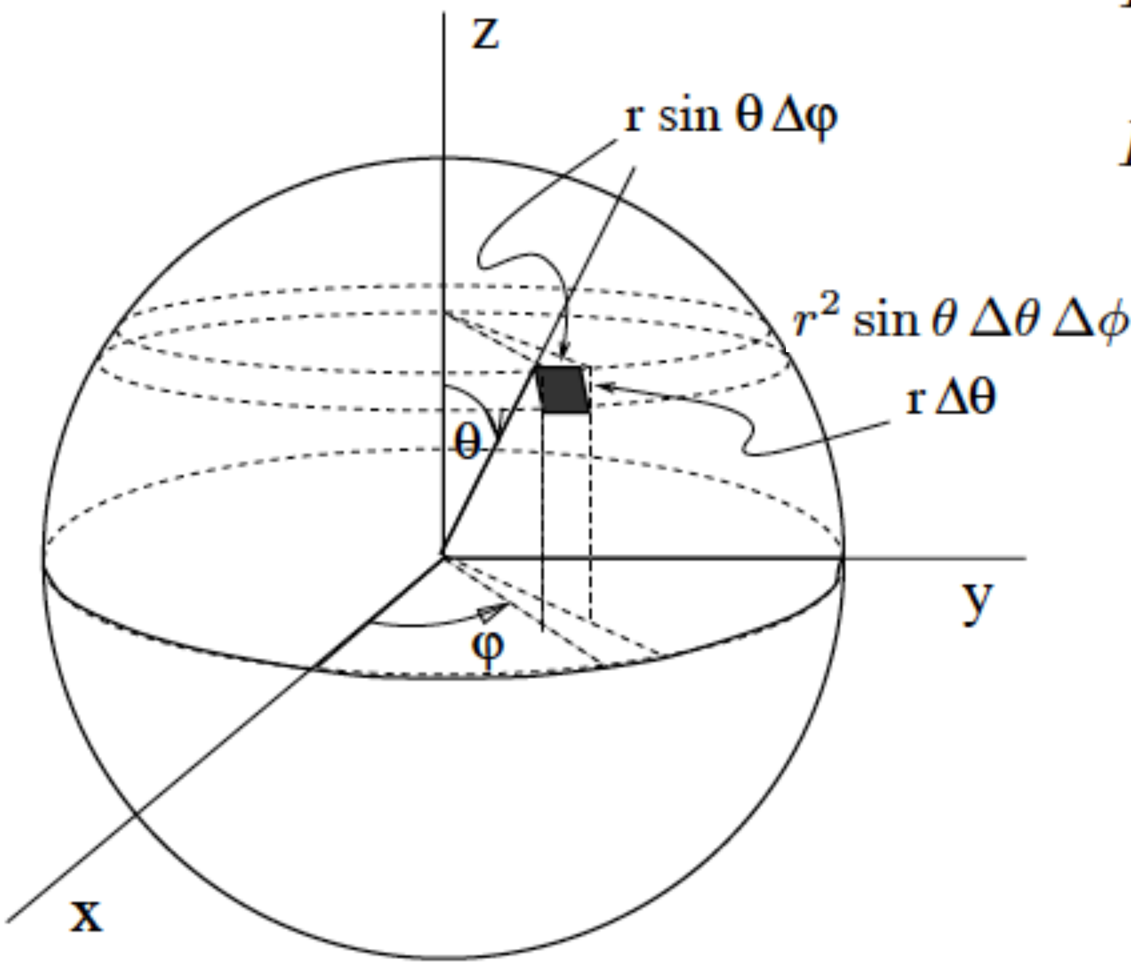
Basic Radiative Transfer

Intensity

$$
\begin{aligned}
\mathrm{d}E_\nu &\equiv I_\nu(\vec{r},\vec{l},t)\,(\vec{l}\cdot\vec{n})\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega \\
&= I_\nu(x,y,z,\theta,\varphi,t)\;\cos\theta\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega.
\end{aligned}
$$

$$I_\lambda = I_\nu\, c/\lambda^2 \qquad \mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}\ ster^{-1}}$$

$$I \equiv \int_0^\infty I_\nu\,\mathrm{d}\nu \qquad \mathrm{W\ m^{-2}\ Hz^{-1}\ ster^{-1}}$$



$$\Delta\Omega = \sin\theta\,\Delta\theta\,\Delta\varphi.$$

Basic Radiative Transfer

Mean Intensity

$$J_\nu(\vec{r}, t) \equiv \frac{1}{4\pi} \int I_\nu \, \mathrm{d}\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I_\nu \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

$\mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}\ ster^{-1}}$

Plane parallel approximation

$$\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta = -2\pi \, \mathrm{d}\mu \qquad \mu \equiv \cos\theta$$

$$J_\nu(z) = \frac{1}{4\pi} \int_0^{\pi} I_\nu(z, \theta) \, 2\pi \sin\theta \, \mathrm{d}\theta = \frac{1}{2} \int_{-1}^{+1} I_\nu(z, \mu) \, \mathrm{d}\mu$$

Basic Radiative Transfer

Flux

$$\mathcal{F}_\nu(\vec{r},\vec{n},t) \equiv \int I_\nu \cos\theta \, \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

$$\mathrm{erg\ s^{-1}\ cm^{-2}\ Hz^{-1}}$$

$$\begin{aligned}
\mathcal{F}_\nu(z) &= \int_0^{2\pi}\int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
&= \int_0^{2\pi}\int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi - \int_0^{2\pi}\int_0^{\pi/2} I_\nu(\pi-\theta) \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
&\equiv \mathcal{F}_\nu^+(z) - \mathcal{F}_\nu^-(z),
\end{aligned}$$

Axial Symmetry

$$\begin{aligned}
\mathcal{F}_\nu(z) &= 2\pi \int_0^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \\
&= 2\pi \int_0^{1} \mu I_\nu \, \mathrm{d}\mu \; - \; 2\pi \int_0^{-1} \mu I_\nu \, \mathrm{d}\mu \\
&= \mathcal{F}_\nu^+(z) \; - \; \mathcal{F}_\nu^-(z).
\end{aligned}$$

Basic Radiative Transfer

Flux

$$\mathcal{F}_\nu^{\text{surface}} \equiv \mathcal{F}_\nu^+(r=R) = \pi \overline{I_\nu^+}$$

$$\mathcal{R}_\nu = \frac{4\pi R^2}{4\pi D^2} \mathcal{F}_\nu^{\text{surface}} = \frac{\pi R^2}{D^2} \overline{I}_\nu$$

Energy Density

$$u_\nu = \frac{1}{c} \int I_\nu \, \mathrm{d}\Omega \qquad \mathrm{erg\,cm^{-3}\,Hz^{-1}}$$

$$u = \int u_\nu \, \mathrm{d}\nu = \frac{1}{c} \iint B_\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu = \frac{4\sigma}{c} T^4$$

$$N_{\text{photon}} = \int_0^\infty \frac{u_\nu}{h\nu} \, \mathrm{d}\nu \approx 20\, T^3 \ \mathrm{cm}^{-3}$$

Basic Radiative Transfer

Pressure

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, \mathrm{d}\Omega$$

dyne $\mathrm{cm}^{-2}\,\mathrm{Hz}^{-1}$

$$p = u/3$$

Moments of Intensity

$$J_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} I_\nu \, \mathrm{d}\mu$$

$$H_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} \mu \, I_\nu \, \mathrm{d}\mu$$

Eddington flux

$$K_\nu(z) \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I_\nu \, \mathrm{d}\mu$$

$$H_\nu = \mathcal{F}_\nu/4\pi = F_\nu/4$$

$$p_\nu = (4\pi/c)\, K_\nu$$

Basic Radiative Transfer

Emission

$$\mathrm{d}E_\nu \equiv j_\nu \, \mathrm{d}V \, \mathrm{d}t \, \mathrm{d}\nu \, \mathrm{d}\Omega \qquad \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}\ ster^{-1}}$$

$$\mathrm{d}I_\nu(s) = j_\nu(s) \, \mathrm{d}s,$$

Extinction

$$\mathrm{d}I_\nu \equiv -\sigma_\nu n \, I_\nu \, \mathrm{d}s$$

$$\mathrm{d}I_\nu \equiv -\alpha_\nu \, I_\nu \, \mathrm{d}s$$

$$\mathrm{d}I_\nu \equiv -\kappa_\nu \rho \, I_\nu \, \mathrm{d}s$$

Basic Radiative Transfer

Source function

$$S_\nu \equiv j_\nu / \alpha_\nu \qquad \text{erg cm}^{-2}\ \text{s}^{-1}\ \text{Hz}^{-1}\ \text{ster}^{-1}$$

$$S_\nu^{\text{tot}} = \frac{\sum j_\nu}{\sum \alpha_\nu}$$

$$S_\nu^{\text{tot}} = \frac{j_\nu^c + j_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{S_\nu^c + \eta_\nu S_\nu^l}{1 + \eta_\nu} \qquad \eta_\nu \equiv \alpha_\nu^l / \alpha_\nu^c$$

Transport Equation

Transport along a ray

$$\mathrm{d}I_\nu(s) = I_\nu(s + \mathrm{d}s) - I_\nu(s) = j_\nu(s)\,\mathrm{d}s - \alpha_\nu(s) I_\nu(s)\,\mathrm{d}s$$

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu$$

$$\frac{\mathrm{d}I_\nu}{\alpha_\nu\,\mathrm{d}s} = S_\nu - I_\nu$$

Optical length and thickness

$$\mathrm{d}\tau_\nu(s) \equiv \alpha_\nu(s)\,\mathrm{d}s$$

$$\tau_\nu(D) = \int_0^D \alpha_\nu(s)\,\mathrm{d}s$$

$$I_\nu(D) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu(D)}$$

Transport Equation

Optical length and thickness

$$< \tau_\nu(s) > \equiv \frac{\int_0^\infty \tau_\nu(s)\, e^{-\tau_\nu(s)}\, d\tau_\nu(s)}{\int_0^\infty e^{-\tau_\nu(s)}\, d\tau_\nu(s)} = 1$$

$$l_\nu = \frac{< \tau_\nu(s) >}{\alpha_\nu} = \frac{1}{\alpha_\nu} = \frac{1}{\kappa_\nu \rho}$$

$$\frac{dI_\nu}{\alpha_\nu\, ds} = S_\nu - I_\nu \qquad \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)\, e^{-(\tau_\nu - t_\nu)}\, dt_\nu$$

Transport Equation

Homogeneous medium

$$I_\nu(\tau_\nu) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(\tau_\nu - t_\nu)}\,\mathrm{d}t_\nu$$

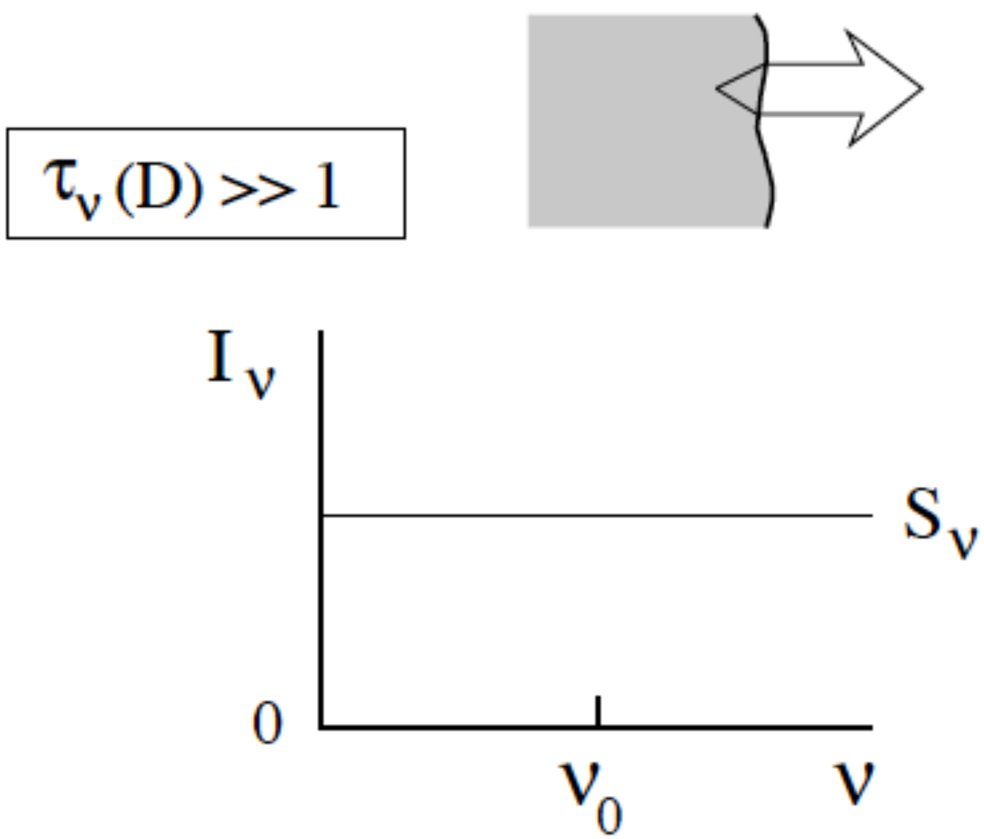
$$I_\nu(D) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu(D)} + S_\nu\left(1 - \mathrm{e}^{-\tau_\nu(D)}\right)$$

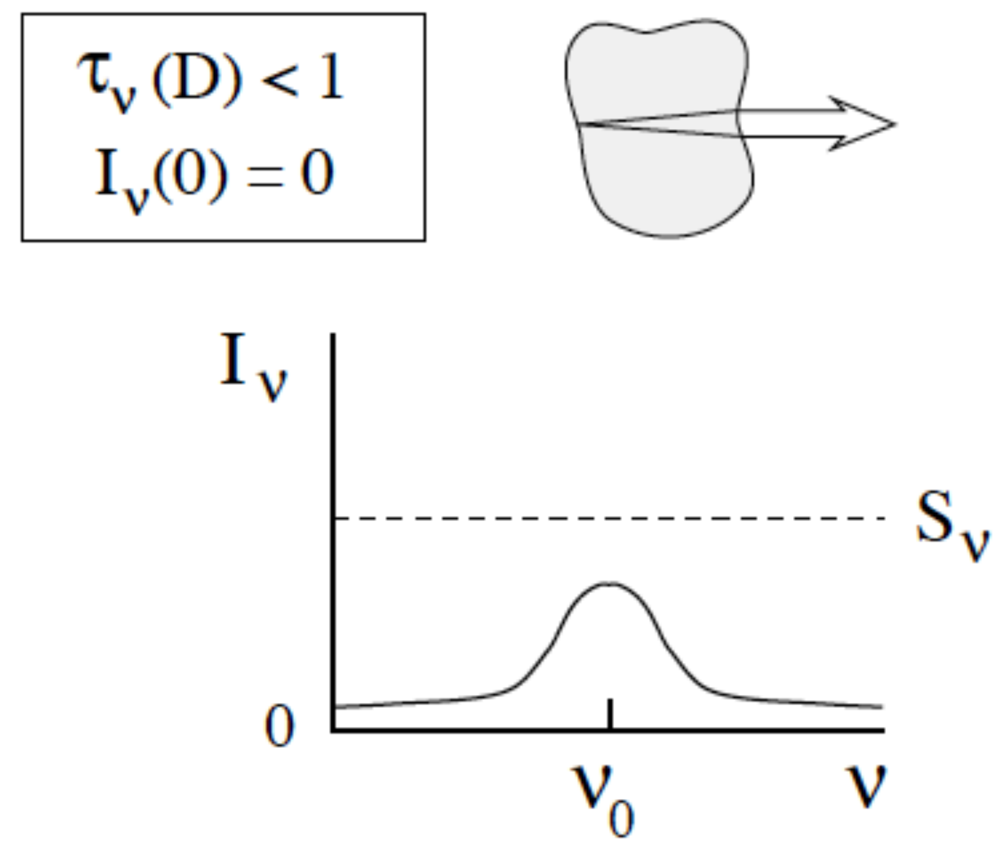
$$I_\nu(D) \approx S_\nu$$ optically thick

$$I_\nu(D) \approx I_\nu(0) + [S_\nu - I_\nu(0)]\,\tau_\nu(D)$$ optically thin

Transport Equation

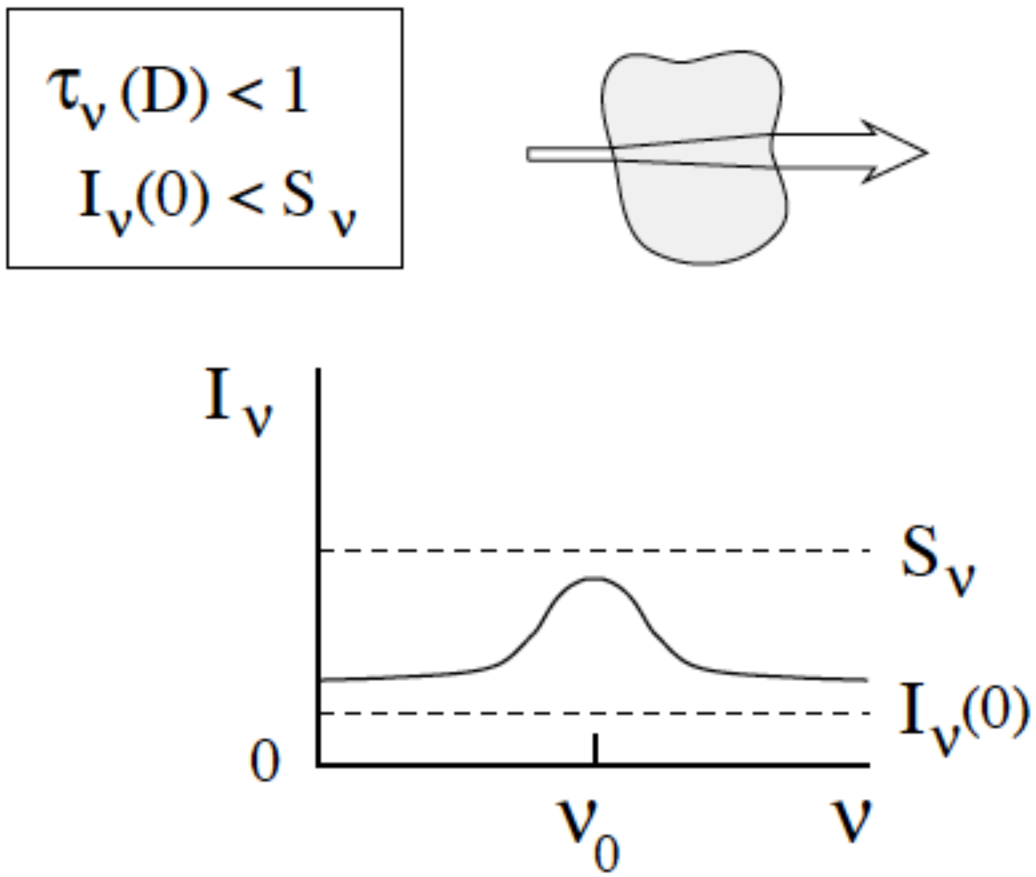
Homogeneous medium

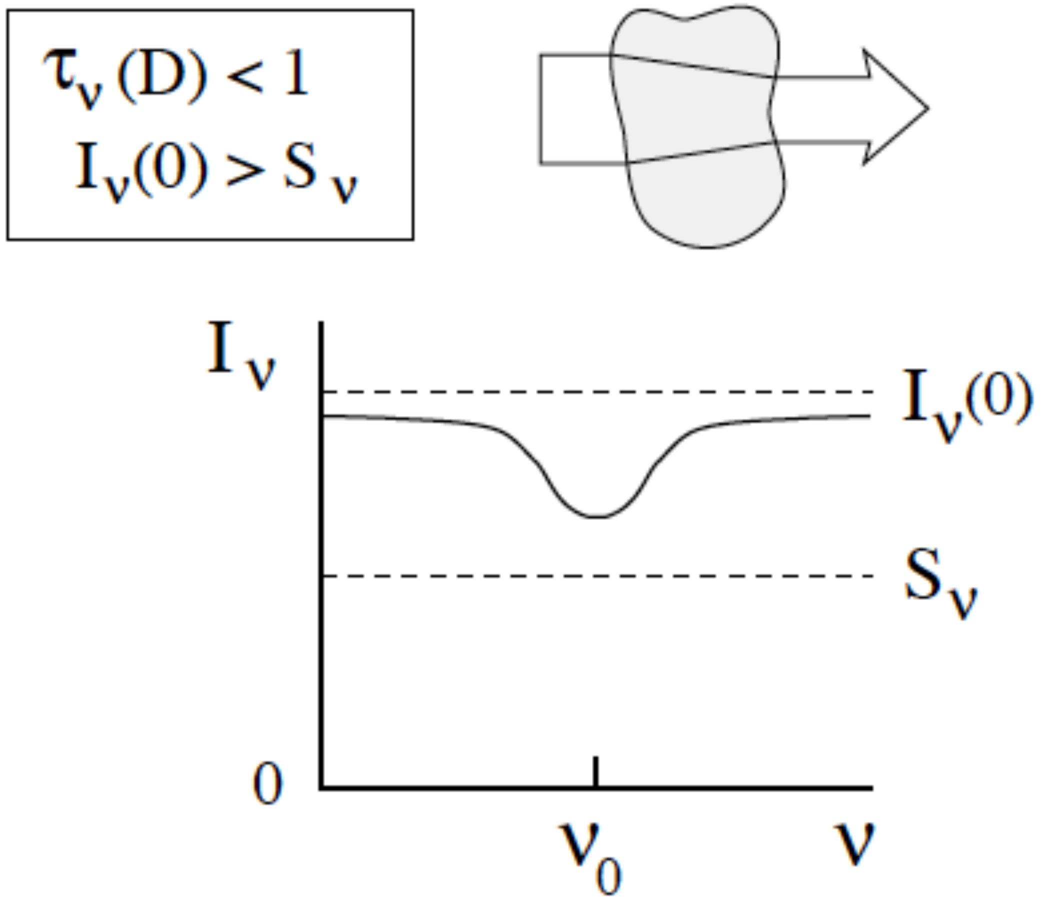




Transport Equation

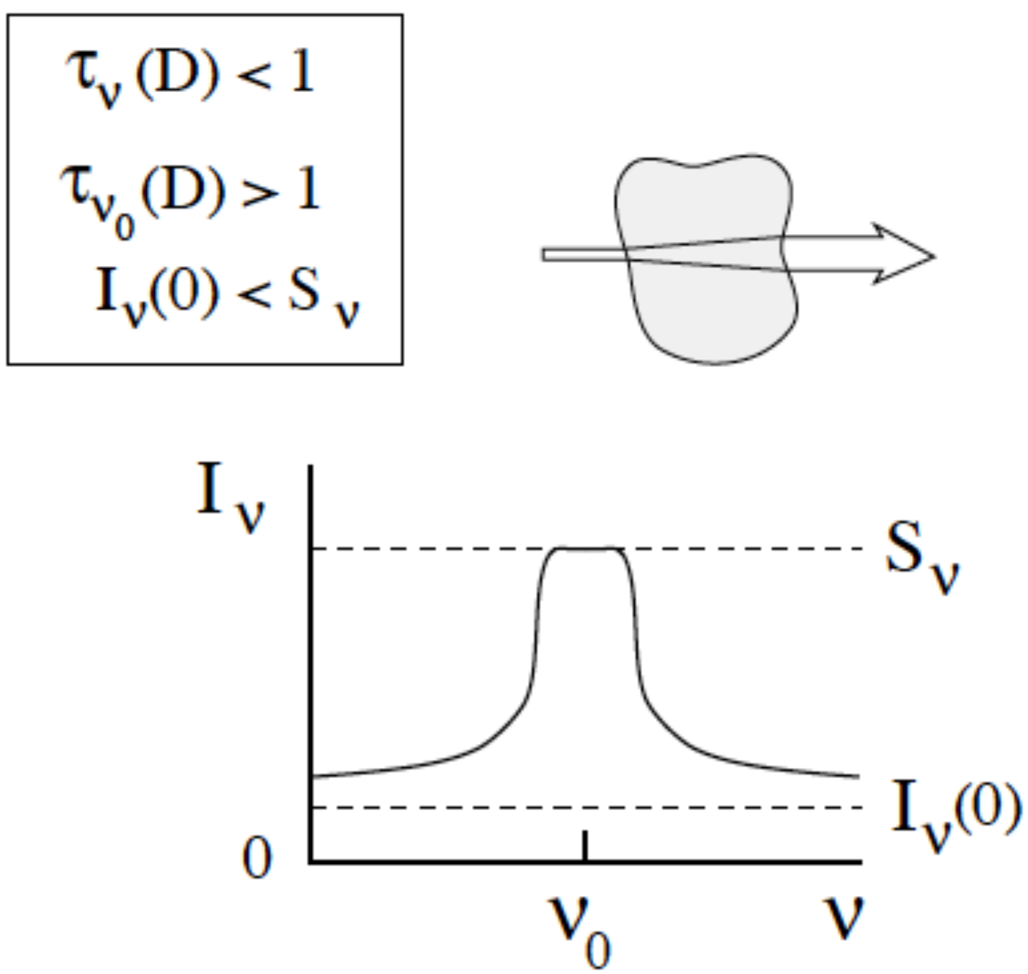
Homogeneous medium

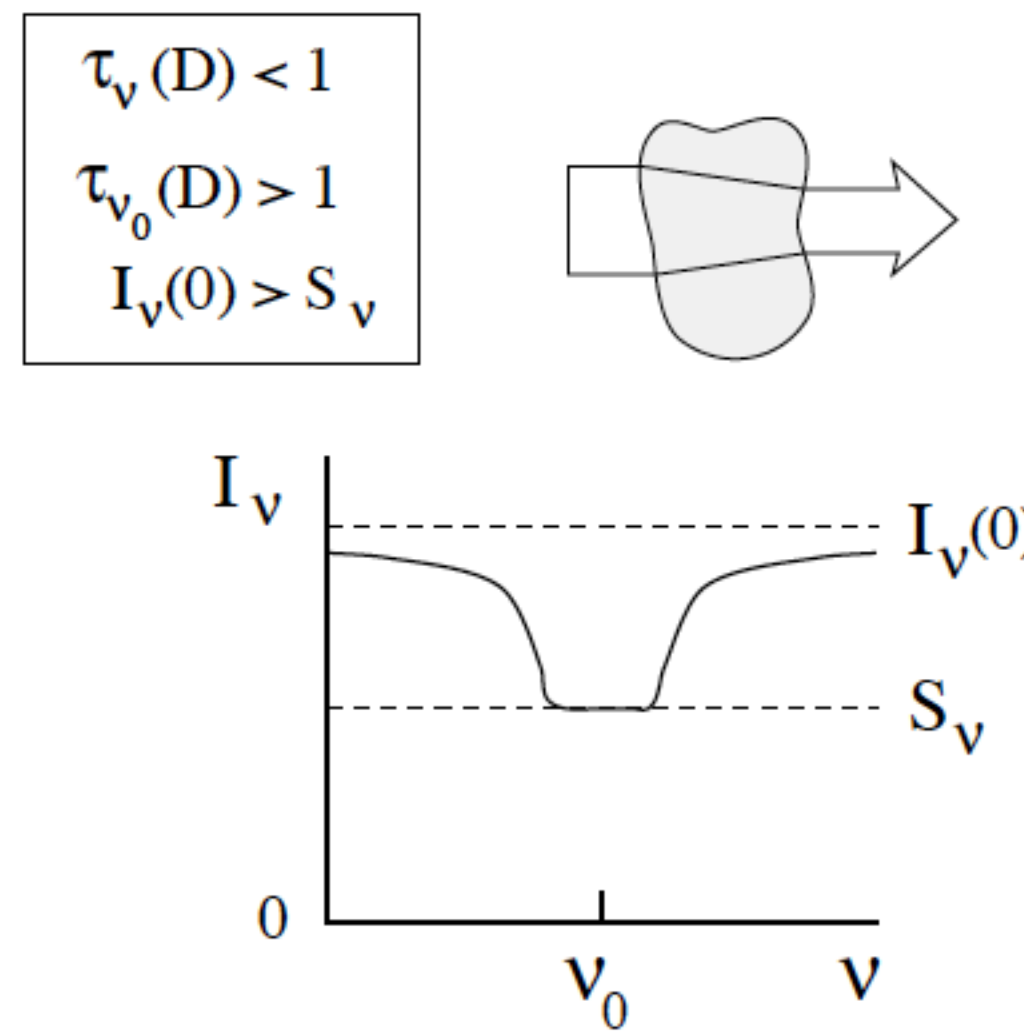




Transport Equation

Homogeneous medium





Transport Equation

Transport through an atmosphere

Optical depth

$$\mathrm{d}\tau_{\nu\mu} \equiv -\alpha_\nu \frac{\mathrm{d}z}{|\mu|} \qquad \mu \equiv \cos\theta$$

$$\tau_\nu(z_0) = \int_\infty^{z_0} -\alpha_\nu \, \mathrm{d}z = \int_{z_0}^{\infty} \alpha_\nu \, \mathrm{d}z$$

$$\mathrm{d}\tau_\nu^{\mathrm{total}} = -(\alpha_\nu^c + \alpha_\nu^l)\, \mathrm{d}z = (1 + \eta_\nu)\, \mathrm{d}\tau_\nu^c \qquad \eta_\nu \equiv \alpha_\nu^l / \alpha_\nu^c$$

Transport Equation

Plane-parallel transport equation

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu \qquad \tau_\nu(z_0) = \int_\infty^{z_0} -\alpha_\nu \, \mathrm{d}z$$

Formal Solution

$\mu < 0$

$$I_\nu^-(\tau_\nu, \mu) = -\int_0^{\tau_\nu} S_\nu(t_\nu) \, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu} \, \mathrm{d}t_\nu/\mu \qquad t_\nu \equiv \int_\infty^z -\alpha_\nu(z) \, \mathrm{d}z$$

$\mu > 0$

$$I_\nu^+(\tau_\nu, \mu) = +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) \, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu} \, \mathrm{d}t_\nu/\mu$$

Transport Equation

Eddington-Barbier approximation

$$I_\nu^+(\tau_\nu, \mu) = +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu}\, \mathrm{d}t_\nu/\mu$$

$\tau_\nu = 0,\ \mu > 0$

$$I_\nu^+(\tau_\nu\!=\!0, \mu) = \int_0^{\infty} S_\nu(t_\nu)\, \mathrm{e}^{-t_\nu/\mu}\, \mathrm{d}t_\nu/\mu$$

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n {\tau_\nu}^n = a_0 + a_1\tau_\nu + a_2{\tau_\nu}^2 + \ldots + a_n{\tau_\nu}^n$$

$$\int_0^\infty x^n \exp(-x)\, \mathrm{d}x = n!$$

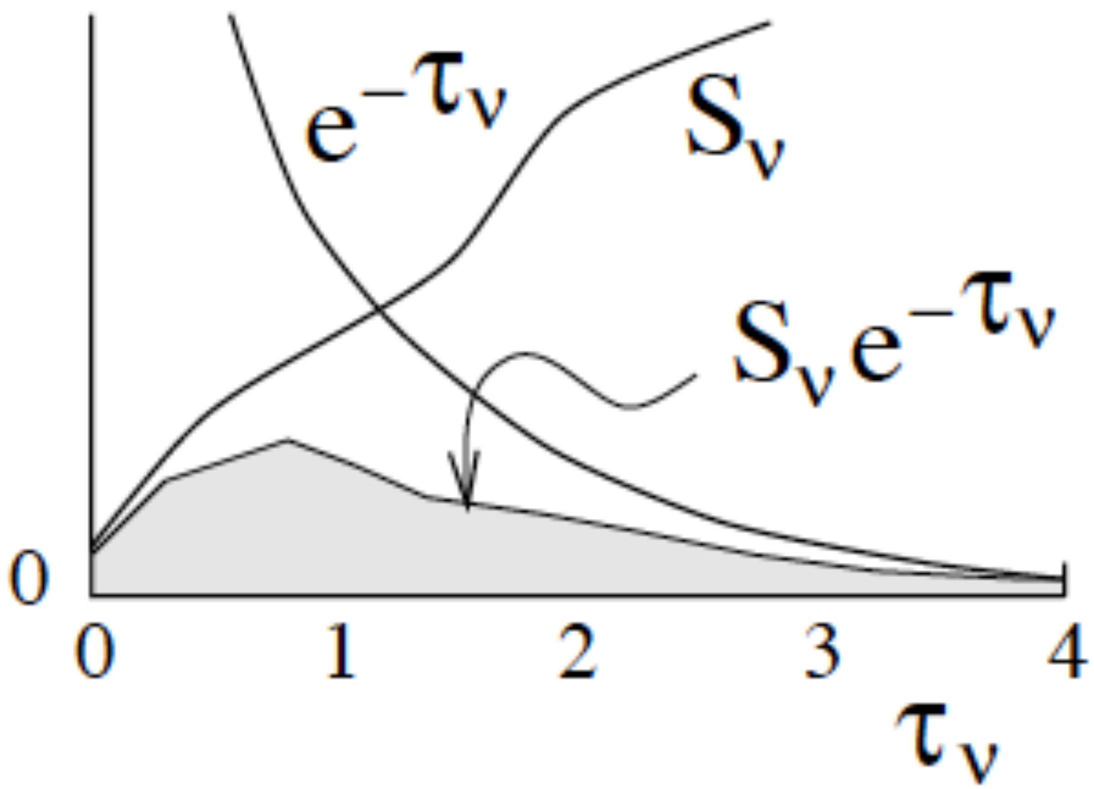
$$I_\nu^+(\tau_\nu\!=\!0, \mu) = a_o + a_1\mu + 2a_2\mu^2 + \ldots + n!\, a_n\mu^n$$

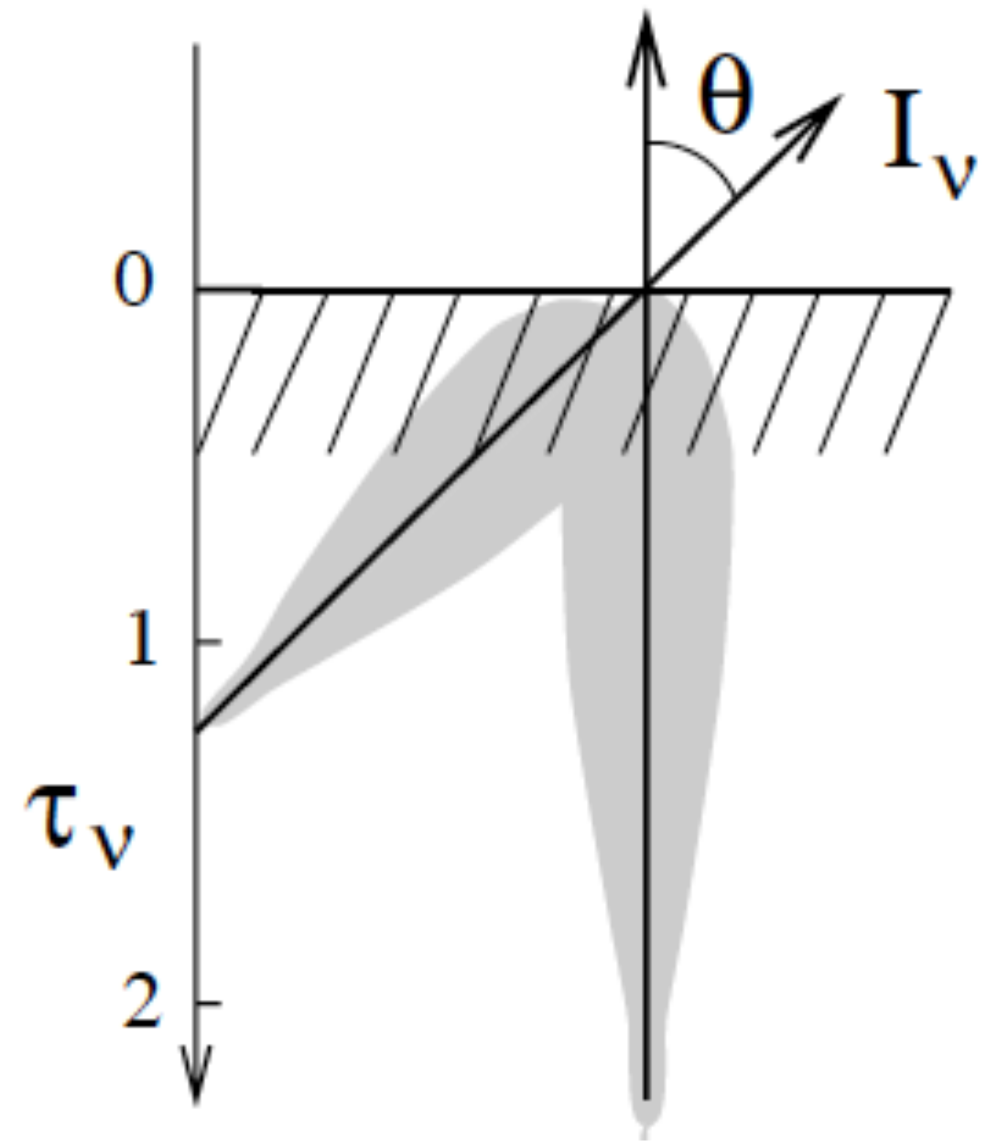
$$I_\nu^+(\tau_\nu\!=\!0, \mu)\ \approx\ S_\nu(\tau_\nu = \mu)$$

$$\mathcal{F}_\nu^+(0) \approx \pi S_\nu(\tau_\nu = 2/3)$$

Transport Equation

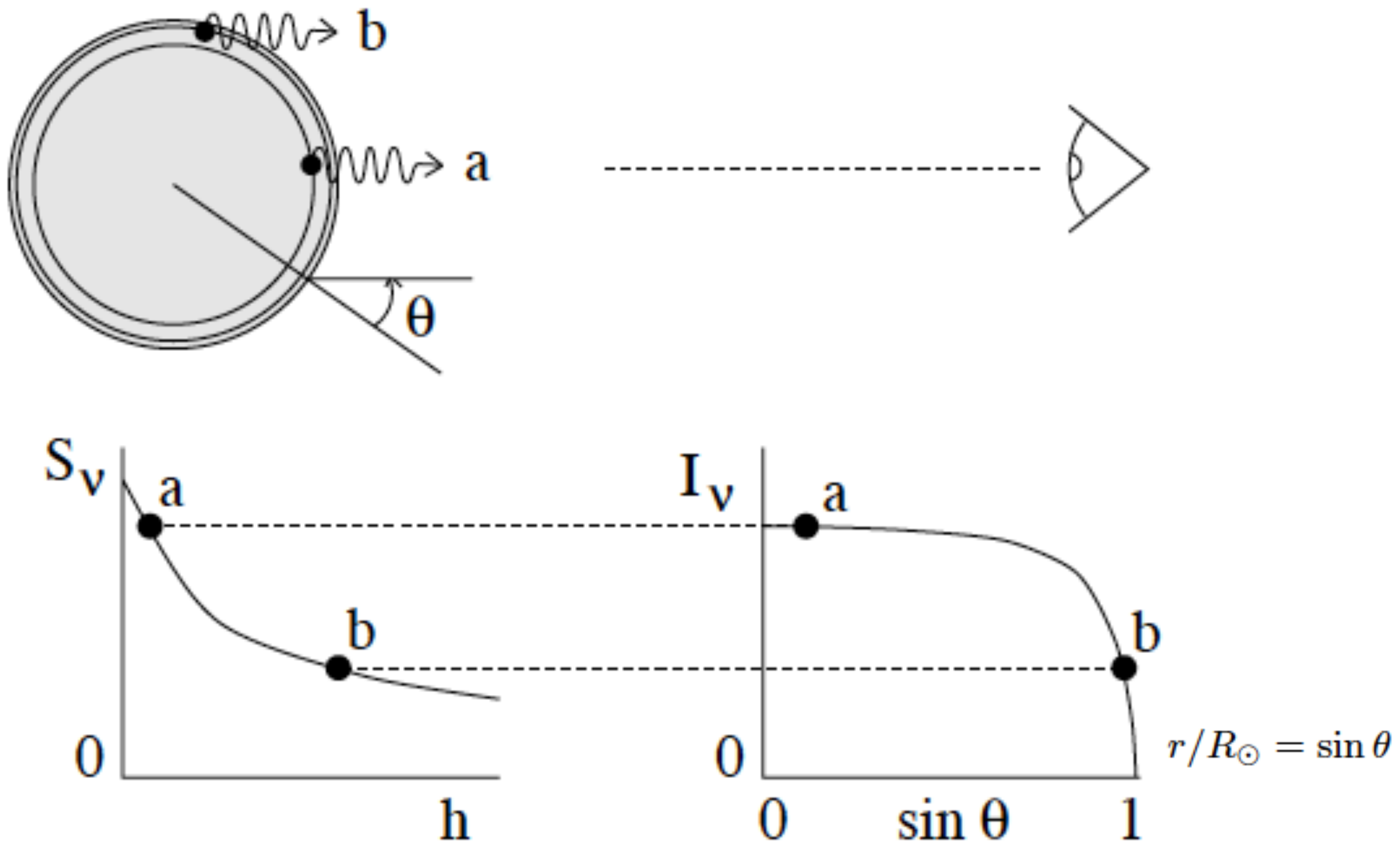
Eddington-Barbier approximation





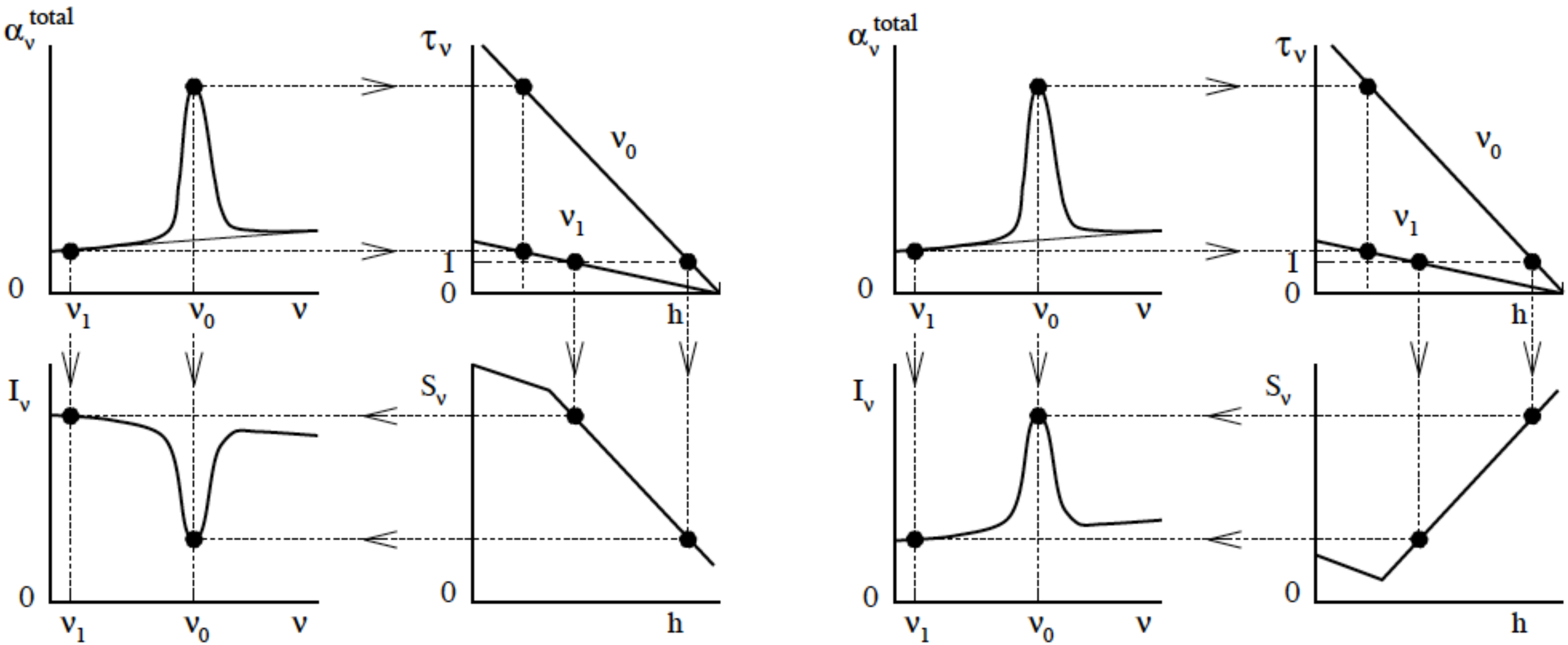
Transport Equation

Eddington-Barbier approximation



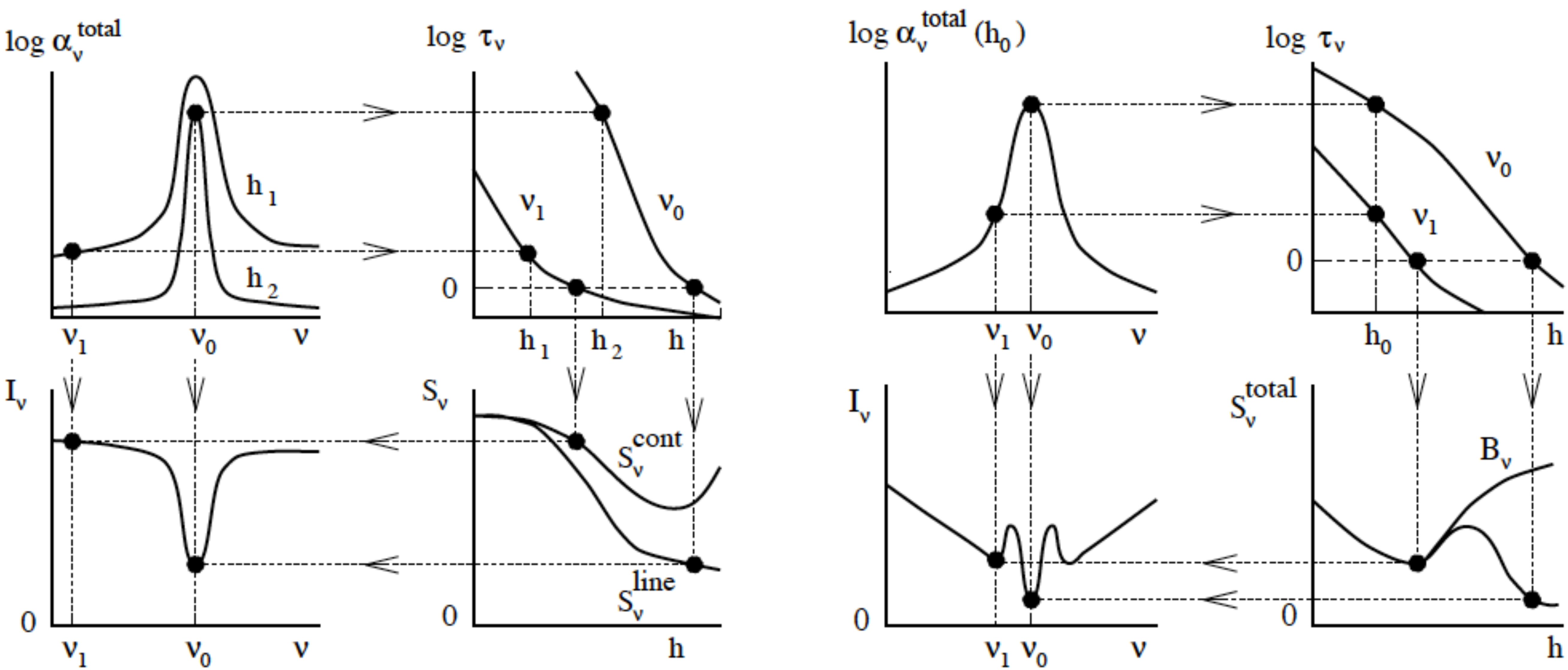
Transport Equation

Eddington-Barbier approximation



Transport Equation

Eddington-Barbier approximation



Line Transitions

Bound - bound transitions

- Radiative excitation
- Spontaneous radiative deexcitation
- Induced radiative deexcitation
- Collisional excitation
- Collisional deexcitation

Line Transitions

Einstein coefficients

Spontaneous deexcitation

$A_{ul} \equiv$ transition probability for spontaneous deexcitation from state u to state l per sec per particle in state u.

$$\Delta t = 1/A_{ul}$$

$$\Delta E = h/(2\pi \Delta t)$$

$$\Delta \nu = \gamma^{\mathrm{rad}}/(2\pi) \qquad \gamma^{\mathrm{rad}} \equiv 1/\Delta t$$

$$\psi(\nu - \nu_0) = \frac{\gamma^{\mathrm{rad}}/4\pi^2}{(\nu - \nu_0)^2 + (\gamma^{\mathrm{rad}}/4\pi)^2}$$

$$\psi(\nu - \nu_0) = \frac{H(a, v)}{\sqrt{\pi}\Delta \nu_{\mathrm{D}}}$$

$$\Delta \nu_{\mathrm{D}} \equiv \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$$

Line Transitions

Einstein coefficients

Radiative excitation

$B_{lu}\overline{J}^{\varphi}_{\nu_0} \equiv$ number of radiative excitations from state l to state u per sec per particle in state l,

$$\overline{J}^{\varphi}_{\nu_0} \equiv \int_0^{\infty} J_\nu \, \varphi(\nu-\nu_0) \, \mathrm{d}\nu \qquad\qquad \int \varphi(\nu-\nu_0)\,\mathrm{d}\nu = 1$$

$$\overline{J}^{\varphi}_{\nu_0} \equiv \frac{1}{2}\int_0^{\infty}\int_{-1}^{+1} I_\nu \, \varphi(\nu-\nu_0) \, \mathrm{d}\mu \, \mathrm{d}\nu$$

$$\varphi(\nu-\nu_0) = \frac{H(a,v)}{\sqrt{\pi}\Delta\nu_{\mathrm{D}}}$$

$$\varphi(\nu=\nu_0) = \frac{1-a}{\sqrt{\pi}\Delta\nu_{\mathrm{D}}} \qquad\qquad a<1$$

Line Transitions

Einstein coefficients

Induced deexcitation

$B_{ul}\overline{J}^{\chi}_{\nu_0} \equiv$ number of induced radiative deexcitations from state u to state l per sec per particle in state u,

$$\overline{J}^{\chi}_{\nu_0} \equiv \frac{1}{2}\int_0^{\infty}\int_{-1}^{+1} I_\nu\, \chi(\nu-\nu_0)\, \mathrm{d}\mu\, \mathrm{d}\nu = \int_0^{\infty} J_\nu \chi(\nu-\nu_0)\, \mathrm{d}\nu$$

Collisional excitation and deexcitation

$C_{lu} \equiv$ number of collisional excitations from state l to state u per sec per particle in state l.

$C_{ul} \equiv$ number of collisional deexcitations from state u to state l per sec per particle in state u

$$n_i C_{ij} = n_i N_\mathrm{e} \int_{v_0}^{\infty} \sigma_{ij}(v)\, v\, f(v)\, \mathrm{d}v \qquad (1/2) m v_0^2 = h\nu_0$$

Line Transitions

Einstein coefficients

Einstein relations

$$\frac{A_{ul}}{B_{ul}} = \frac{2h\nu^3}{c^2}$$

$$\frac{B_{lu}}{B_{ul}} = \frac{g_u}{g_l}$$

$$\frac{C_{ul}}{C_{lu}} = \frac{g_l}{g_u}\,\mathrm{e}^{E_{ul}/kT}$$

Line Transitions

Volume coefficients

Extinction

$$\alpha^l_\nu = \frac{h\nu}{4\pi}\left[n_l B_{lu}\,\varphi(\nu-\nu_0) - n_u B_{ul}\,\chi(\nu-\nu_0)\right]$$
$$= \frac{h\nu}{4\pi} n_l B_{lu}\,\varphi(\nu-\nu_0)\left[1 - \frac{n_u\, g_l\, \chi(\nu-\nu_0)}{n_l\, g_u\, \varphi(\nu-\nu_0)}\right]$$

$$\alpha^l_{\nu_0} \equiv \int_0^\infty \alpha^l_\nu \,\mathrm{d}\nu = \frac{h\nu_0}{4\pi}\,(n_l B_{lu} - n_u B_{ul})$$

$$\sigma^l_\nu = \frac{h\nu}{4\pi} B_{lu}\,\varphi(\nu-\nu_0)$$

$$\sigma^l_{\nu_0} \equiv \int_0^\infty \sigma^l_\nu \,\mathrm{d}\nu = \frac{h\nu_0}{4\pi} B_{lu} = \frac{\pi e^2}{m_e c} f_{lu} = 0.02654\, f_{lu} \quad \mathrm{cm}^2\,\mathrm{Hz}$$

$$A_{ul} \sim \frac{g_l}{g_u} f_{lu}\,(\Delta E_{ul})^2 \qquad\qquad A_{ul} = 6.67\times 10^{13}\,\frac{g_l}{g_u}\,\frac{f_{lu}}{\lambda^2} \quad \mathrm{s}^{-1}$$

Line Transitions

Volume coefficients

Emission

$$j_\nu^l = \frac{h\nu}{4\pi} n_u A_{ul}\, \psi(\nu - \nu_0) \qquad j_{\nu_0}^l = \int_0^\infty j_\nu^l \, \mathrm{d}\nu = \frac{h\nu_0}{4\pi} n_u A_{ul}$$

Source function

$$S_\nu^l \equiv j_\nu^l / \alpha_\nu^l = \frac{n_u A_{ul} \psi(\nu - \nu_0)}{n_l B_{lu} \varphi(\nu - \nu_0) - n_u B_{ul} \chi(\nu - \nu_0)}$$

$$S_\nu^l = \frac{\frac{A_{ul}}{B_{ul}} \frac{\psi}{\varphi}}{\frac{n_l}{n_u} \frac{B_{lu}}{B_{ul}} - \frac{\chi}{\varphi}} = \frac{2h\nu^3}{c^2} \frac{\psi/\varphi}{\frac{g_u n_l}{g_l n_u} - \frac{\chi}{\varphi}}.$$

$$\varphi(\nu - \nu_0) = \psi(\nu - \nu_0) = \chi(\nu - \nu_0)$$

$$S_{\nu_0}^l = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}} = \frac{2h\nu_0^3}{c^2} \frac{1}{\frac{g_u n_l}{g_l n_u} - 1}.$$

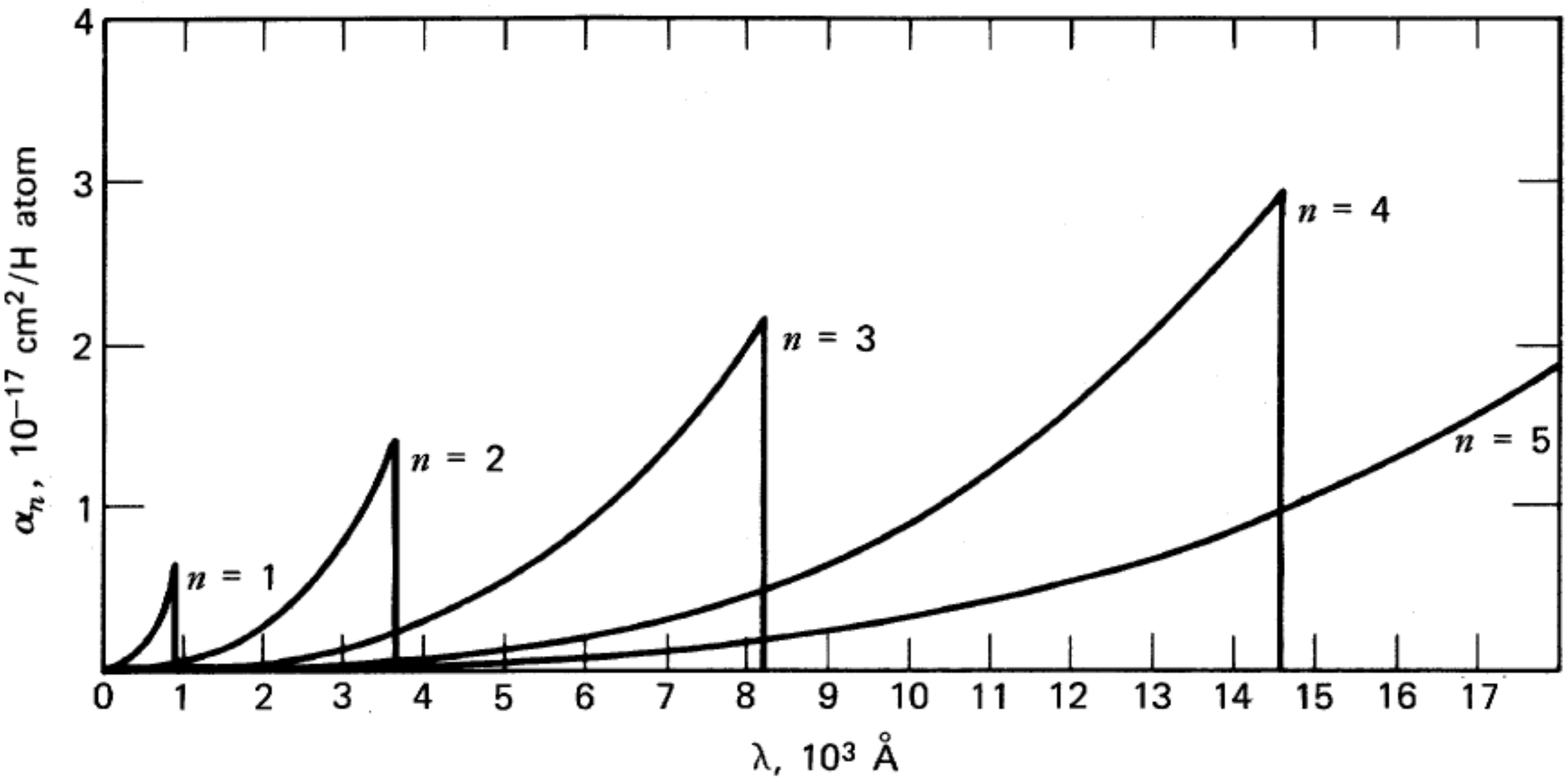
Continuum Transitions

Inelastic Processes

Bound-free transitions: Kramer's formula

$$\sigma_\nu^{\mathrm{bf}} = 2.815 \times 10^{29} \frac{Z^4}{n^5 \nu^3} g_{\mathrm{bf}} \qquad \nu \geq \nu_0$$

$$\alpha_\nu^{\mathrm{bf}} = \sigma_\nu^{\mathrm{bf}} n_i \left(1 - \mathrm{e}^{-h\nu/kT}\right)$$



Continuum Transitions

Inelastic Processes

Free-free transitions

$$S_\nu = B_\nu$$

$$\sigma_\nu^{\mathrm{ff}} = 3.7 \times 10^8 \, N_\mathrm{e} \frac{Z^2}{T^{1/2} \nu^3} \, g_{\mathrm{ff}}$$

$$\alpha_\nu^{\mathrm{ff}} = \sigma_\nu^{\mathrm{ff}} \, N_{\mathrm{ion}} \left(1 - \, \mathrm{e}^{-h\nu/kT}\right)$$

Wien limit

$$\alpha_\nu^{\mathrm{ff}} \approx 3.7 \times 10^8 \, N_\mathrm{e} N_{\mathrm{ion}} \frac{Z^2}{T^{1/2} \nu^3} \, g_{\mathrm{ff}}$$

Rayleigh-Jeans limit

$$\alpha_\nu^{\mathrm{ff}} \approx 0.018 \, N_\mathrm{e} N_{\mathrm{ion}} \frac{Z^2}{T^{3/2} \nu^2} \, g_{\mathrm{ff}}$$

Continuum Transitions

Elastic Processes

Thomson scattering

$$\sigma_\nu^{\mathrm{T}} \equiv \sigma^{\mathrm{T}} = \frac{8\pi}{3} r_{\mathrm{e}}^2 = 6.65 \times 10^{-25} \ \mathrm{cm}^2$$

$$\alpha_\nu^{\mathrm{T}} = \sigma^{\mathrm{T}} N_{\mathrm{e}}$$

Rayleigh scattering

$$\sigma_\nu^{\mathrm{R}} \approx f_{lu} \, \sigma^{\mathrm{T}} \left(\frac{\nu}{\nu_0} \right)^4 \qquad \nu \ll \nu_0$$

$$\alpha_\nu^{\mathrm{R}} = \sigma_\nu^{\mathrm{R}} N_{\mathrm{H}}$$

Local Thermodynamic Equilibrium

$$S^l_{\nu_0} = B_{\nu_0}$$

Matter in LTE

Maxwell distribution

$$\left[\frac{n(v_x)}{N}\,\mathrm{d}v_x\right]_{\mathrm{LTE}} = \left(\frac{m}{2\pi kT}\right)^{1/2} \mathrm{e}^{-(1/2)mv_x^2/kT}\,\mathrm{d}v_x$$

$$\left[\frac{n(v)}{N}\,\mathrm{d}v\right]_{\mathrm{LTE}} = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2\,\mathrm{e}^{-(1/2)mv^2/kT}\,\mathrm{d}v$$

$$v_p = \sqrt{2kT/m}$$

$$\langle v \rangle = \sqrt{3kT/m}.$$

Local Thermodynamic Equilibrium

Matter in LTE

Boltzmann distribution

$$\left[\frac{n_{r,s}}{n_{r,t}}\right]_{\mathrm{LTE}} = \frac{g_{r,s}}{g_{r,t}}\,\mathrm{e}^{-(\chi_{r,s}-\chi_{r,t})/kT}$$

Saha distribution

$$\left[\frac{n_{r+1,1}}{n_{r,1}}\right]_{\mathrm{LTE}} = \frac{1}{N_\mathrm{e}}\,\frac{2\,g_{r+1,1}}{g_{r,1}}\left(\frac{2\pi m_\mathrm{e} kT}{h^2}\right)^{3/2}\mathrm{e}^{-\chi_r/kT} \qquad \chi_r = h\nu_{\mathrm{threshold}}$$

$$\left[\frac{N_{r+1}}{N_r}\right]_{\mathrm{LTE}} = \frac{1}{N_\mathrm{e}}\,\frac{2\,U_{r+1}}{U_r}\left(\frac{2\pi m_\mathrm{e} kT}{h^2}\right)^{3/2}\mathrm{e}^{-\chi_r/kT}$$

$$U_r \equiv \sum_s g_{r,s}\,\mathrm{e}^{-\chi_{r,s}/kT}$$

Local Thermodynamic Equilibrium

Matter in LTE

Saha-Boltzmann distribution

$$\left[\frac{n_c}{n_i}\right]_{\mathrm{LTE}} = \frac{1}{N_{\mathrm{e}}} \frac{2\, g_c}{g_i} \left(\frac{2\pi m_{\mathrm{e}} kT}{h^2}\right)^{3/2} \mathrm{e}^{-\chi_{ci}/kT}$$

$$\chi_{ci} = \chi_r - \chi_{r,i} + \chi_{r+1,c} = h\nu_{\mathrm{threshold}}$$

Local Thermodynamic Equilibrium

Radiation in LTE

Planck function

$$\left[S_\nu^l\right]_{\rm LTE} = \frac{2h\nu^3}{c^2}\frac{1}{\left[\dfrac{g_u n_l}{g_l n_u}\right]_{\rm LTE} - 1}$$

$$= \frac{2h\nu^3}{c^2}\frac{1}{{\rm e}^{h\nu/kT} - 1} \equiv B_\nu(T)$$

Wien and Rayleigh-Jeans approximations

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\,{\rm e}^{-h\nu/kT} \qquad \exp(h\nu/kT) \gg 1$$

$$B_\nu(T) \approx \frac{2\nu^2 kT}{c^2} \qquad \exp(h\nu/kT) - 1 \approx h\nu/kT$$

Local Thermodynamic Equilibrium

Radiation in LTE

Stefan-Boltzmann law

$$B(T) = \int_0^\infty B_\nu \, \mathrm{d}\nu = \frac{\sigma}{\pi} T^4$$

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-5} \quad \mathrm{erg\,cm^{-2}\,K^{-4}\,s^{-1}}$$

Induced emission

$$\left[1 - \frac{n_u B_{ul} \chi(\nu - \nu_0)}{n_l B_{lu} \varphi(\nu - \nu_0)}\right]_{\mathrm{LTE}} = 1 - \, \mathrm{e}^{-h\nu_0/kT}$$

Line extinction

$$\left[\alpha_\nu^l\right]_{\mathrm{LTE}} = \frac{\pi e^2}{m_e c} n_l^{\mathrm{LTE}} \, f_{lu} \, \varphi(\nu - \nu_0) \left[1 - \, \mathrm{e}^{-h\nu_0/kT}\right]$$

$$S_\nu^l(\vec{r}) = B_\nu \left[T(\vec{r})\right] \qquad I_\nu(\vec{r}, \vec{l}) \neq B_\nu \left[T(\vec{r})\right] \qquad J_\nu(\vec{r}) \neq B_\nu \left[T(\vec{r})\right] \qquad \mathcal{F}_\nu(\vec{r}) \neq 0$$

Non-local Thermodynamic Equilibrium

Statistical equilibrium

Rate equations

$$\frac{\mathrm{d}n_i(\vec{r})}{\mathrm{d}t} = \sum_{j\neq i}^{N} n_j(\vec{r})P_{ji}(\vec{r}) - n_i(\vec{r})\sum_{j\neq i}^{N} P_{ij}(\vec{r}) = 0$$

$$P_{ij} = R_{ij} + C_{ij}$$

$$R_{ij} = A_{ij} + B_{ij}\overline{J}_{\nu_0}$$

Transport equations

$$\mu\,\frac{\mathrm{d}I_\nu(\vec{r},\mu)}{\mathrm{d}\tau_\nu(\vec{r})} = -S_\nu(\vec{r}) + I_\nu(\vec{r},\mu)$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Departure coefficients

$$b_l = n_l/n_l^{\mathrm{LTE}} \qquad b_u = n_u/n_u^{\mathrm{LTE}}$$

Bound-bound source function

$$S_\nu^l = \frac{2h\nu^3}{c^2}\frac{\psi/\varphi}{\dfrac{b_l}{b_u}\,\mathrm{e}^{h\nu/kT} - \dfrac{\chi}{\varphi}}$$

$$S_{\nu_0}^l = \frac{2h\nu_0^3}{c^2}\frac{1}{\dfrac{b_l}{b_u}\,\mathrm{e}^{h\nu_0/kT} - 1} \qquad \chi_\nu = \psi_\nu = \varphi_\nu$$

$$S_{\nu_0}^l \approx \frac{b_u}{b_l} B_{\nu_0} \qquad (b_l/b_u)\exp(h\nu/kT) \gg 1$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Bound-bound extinction

$$\begin{aligned}
\alpha_\nu^l &= \frac{h\nu}{4\pi} b_l\, n_l^{\mathrm{LTE}} B_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u\, n_u^{\mathrm{LTE}}\, B_{ul}\, \chi}{b_l\, n_l^{\mathrm{LTE}}\, B_{lu}\, \varphi}\right] \\
&= \frac{h\nu}{4\pi} b_l\, n_l^{\mathrm{LTE}} B_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u}{b_l} \frac{\chi}{\varphi}\, \mathrm{e}^{-h\nu/kT}\right] \\
&= b_l\, n_l^{\mathrm{LTE}} \sigma_\nu^l \left[1 - \frac{b_u}{b_l} \frac{\chi}{\varphi}\, \mathrm{e}^{-h\nu/kT}\right] \\
&= \frac{\pi e^2}{m_{\mathrm{e}} c} b_l\, n_l^{\mathrm{LTE}} f_{lu} \varphi(\nu-\nu_0) \left[1 - \frac{b_u}{b_l} \frac{\chi}{\varphi}\, \mathrm{e}^{-h\nu/kT}\right]
\end{aligned}$$

$$\alpha_\nu^l \approx b_l \left[\alpha_\nu^l\right]_{\mathrm{LTE}} \qquad \chi/\varphi = 1$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Bound-bound extinction

$$
\begin{aligned}
\alpha^l_{\nu_0} &= \frac{h\nu_0}{4\pi}\, b_l\, n_l^{\rm LTE} B_{lu} \left[1 - \frac{b_u}{b_l}\, {\rm e}^{-h\nu_0/kT}\right] \\
&= \frac{\pi e^2}{m_{\rm e} c}\, b_l\, n_l^{\rm LTE} f_{lu} \left[1 - \frac{b_u}{b_l}\, {\rm e}^{-h\nu_0/kT}\right] \\
&\approx b_l \left[\alpha^l_{\nu_0}\right]_{\rm LTE} .
\end{aligned}
$$

$$
j^l_\nu = b_u\, [\alpha^l_\nu]_{\rm LTE} B_\nu
$$

Non-local Thermodynamic Equilibrium

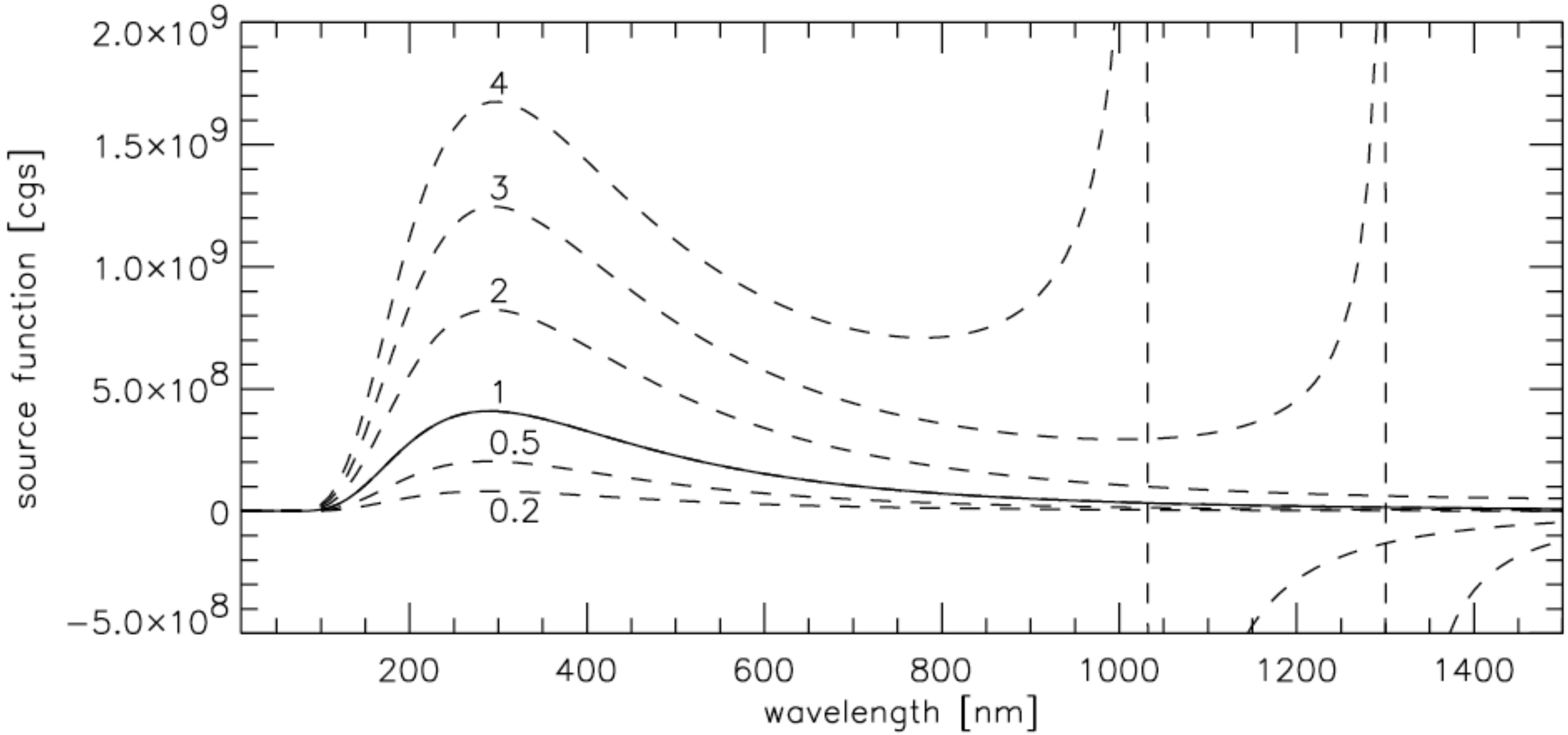
NLTE descriptions

Laser regime

$$\frac{S^l_{\nu_0}}{B_{\nu_0}} = \frac{1 - \ \mathrm{e}^{-h\nu_0/kT}}{(b_l/b_u)\left[1 - (b_u/b_l)\,\mathrm{e}^{-h\nu_0/kT}\right]} = b_u \frac{\left[\alpha^l_{\nu_0}\right]_{\mathrm{LTE}}}{\alpha^l_{\nu_0}}$$

$h\nu \ll kT$

$b_u > b_l$



Non-local Thermodynamic Equilibrium

NLTE descriptions

Bound-free source function

$$S_\nu^{\mathrm{bf}} = \frac{2h\nu^3}{c^2} \frac{1}{\frac{b_i}{b_c}\,\mathrm{e}^{h\nu/kT} - 1}$$

$$S_\nu^{\mathrm{bf}} \approx \frac{b_c}{b_i} B_\nu$$

Bound-free extinction

$$\alpha_\nu^{\mathrm{bf}} = b_i\, n_i^{\mathrm{LTE}} \sigma_{ic}(\nu) \left(1 - \frac{b_c}{b_i}\,\mathrm{e}^{-h\nu/kT}\right)$$

Bund-free emission

$$j_\nu^{\mathrm{bf}} = \alpha_\nu^{\mathrm{bf}}\, S_\nu^{\mathrm{bf}} = b_c \left[\alpha_\nu^{\mathrm{bf}}\right]_{\mathrm{LTE}} B_\nu$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Free-free source function, extinction and emission

$$
\begin{aligned}
S_\nu^{\mathrm{ff}} &= B_\nu \\
\alpha_\nu^{\mathrm{ff}} &= b_c\, n_c^{\mathrm{LTE}} \sigma_\nu^{\mathrm{ff}} \left(1 - \mathrm{e}^{-h\nu/kT}\right) \\
j_\nu^{\mathrm{ff}} &= b_c \left[\alpha_\nu^{\mathrm{ff}}\right]_{\mathrm{LTE}} B_\nu
\end{aligned}
$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Formal temperatures

Excitation temperature

$$\frac{n_u}{n_l} \equiv \frac{g_u}{g_l}\, \mathrm{e}^{-h\nu/kT_{\mathrm{exc}}}$$

$$S^l_{\nu_0} = \frac{2h\nu_0^3}{c^2} \frac{1}{\frac{g_u n_l}{g_l n_u} - 1} = \frac{2h\nu_0^3}{c^2} \frac{1}{\mathrm{e}^{h\nu_0/kT_{\mathrm{exc}}} - 1} = B_{\nu_0}(T_{\mathrm{exc}})$$

Ionisation temperature

$$S^{\mathrm{bf}}_{\nu} \equiv \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/kT_{\mathrm{ion}}} - 1} = B_{\nu}(T_{\mathrm{ion}})$$

Non-local Thermodynamic Equilibrium

NLTE descriptions

Formal temperatures

Radiation temperature

$$B_\nu(T_{\rm rad}) \equiv J_\nu$$

Brightness temperature

$$B_\nu(T_{\rm b}) \equiv I_\nu$$

$$T_{\rm b} \;=\; T_{\rm e}(\tau_\nu = \mu)$$

Effective temperature

$$\pi B(T_{\rm eff}) = \sigma T_{\rm eff}^4 \equiv \mathcal{F}_{\rm surface}$$

Non-local Thermodynamic Equilibrium

Coherent scattering

Two-levels atoms

- Photon scattering
- Photon creation
- Photon destruction

Coherently scattering medium

$$\alpha_\nu^l = \alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}$$

Destruction probability

$$\varepsilon_\nu \equiv \frac{\alpha_\nu^{\mathrm{a}}}{\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}}$$

$$1 - \varepsilon_\nu = \frac{\alpha_\nu^{\mathrm{s}}}{\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}}$$

Non-local Thermodynamic Equilibrium

Coherent scattering

Effective path, thickens, depth

$$l_\nu^* \approx \sqrt{N}\, l_\nu$$

$$l_\nu = \frac{\langle \tau_\nu \rangle}{\alpha_\nu} = \frac{1}{\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}}$$

$$N = 1/\varepsilon_\nu$$

$$l_\nu^* \approx l_\nu / \sqrt{\varepsilon_\nu}$$

$$\tau_\nu^* = \sqrt{\varepsilon_\nu}\, \tau_\nu$$

$$\mathrm{d}\tau_\nu^* = \sqrt{\varepsilon_\nu}\, \mathrm{d}\tau_\nu$$

Non-local Thermodynamic Equilibrium

Coherent scattering

Source function

$$j_\nu^{\mathrm{a}} = \alpha_\nu^{\mathrm{a}} B_\nu$$

$$j_\nu^{\mathrm{s}} = \alpha_\nu^{\mathrm{s}} J_\nu$$

$$S_\nu^l = \frac{j_\nu^{\mathrm{a}} + j_\nu^{\mathrm{s}}}{\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}} = (1 - \varepsilon_\nu)\, J_\nu + \varepsilon_\nu B_\nu$$

$$S_{\nu_0}^l = (1 - \varepsilon_{\nu_0})\, \overline{J}_{\nu_0}^{\varphi} + \varepsilon_{\nu_0} B_{\nu_0}$$

$$\varepsilon_{\nu_0} \equiv \frac{\alpha_{\nu_0}^{\mathrm{a}}}{\alpha_{\nu_0}^{\mathrm{a}} + \alpha_{\nu_0}^{\mathrm{s}}}$$

Non-local Thermodynamic Equilibrium

Coherent scattering

Transport equation

$$\mathrm{d}I_\nu = -\alpha_\nu^{\mathrm{a}} I_\nu \,\mathrm{d}s - \alpha_\nu^{\mathrm{s}} I_\nu \,\mathrm{d}s + \alpha_\nu^{\mathrm{a}} B_\nu \,\mathrm{d}s + \alpha_\nu^{\mathrm{s}} J_\nu \,\mathrm{d}s$$

$$\mathrm{d}\tau_\nu \equiv \alpha_\nu^{l} \,\mathrm{d}s = (\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}) \,\mathrm{d}s$$

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathrm{d}I_\nu}{(\alpha_\nu^{\mathrm{a}} + \alpha_\nu^{\mathrm{s}}) \,\mathrm{d}s} = S_\nu^{l} - I_\nu$$

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu^{l}$$

$$S_\nu^{l} = S_{\nu_0}^{l}$$

Stellar Atmospheres

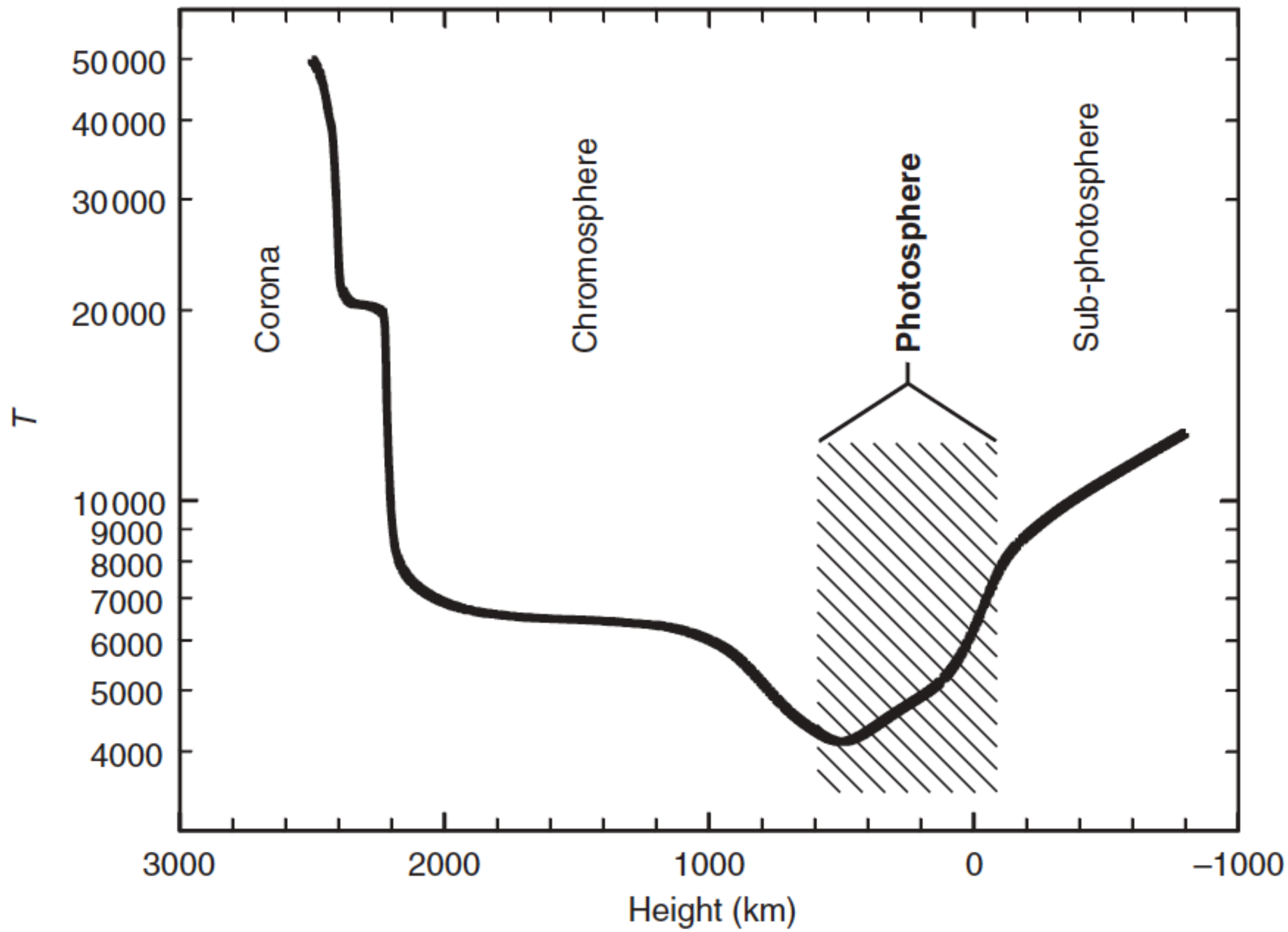
Ricardo Chávez Murillo

May 2021

Introduction to Stellar Atmospheres

What is a Stellar Atmosphere?

- It is a transition region from the stellar interior to the interstellar medium.



Introduction to Stellar Atmospheres

Stellar photosphere

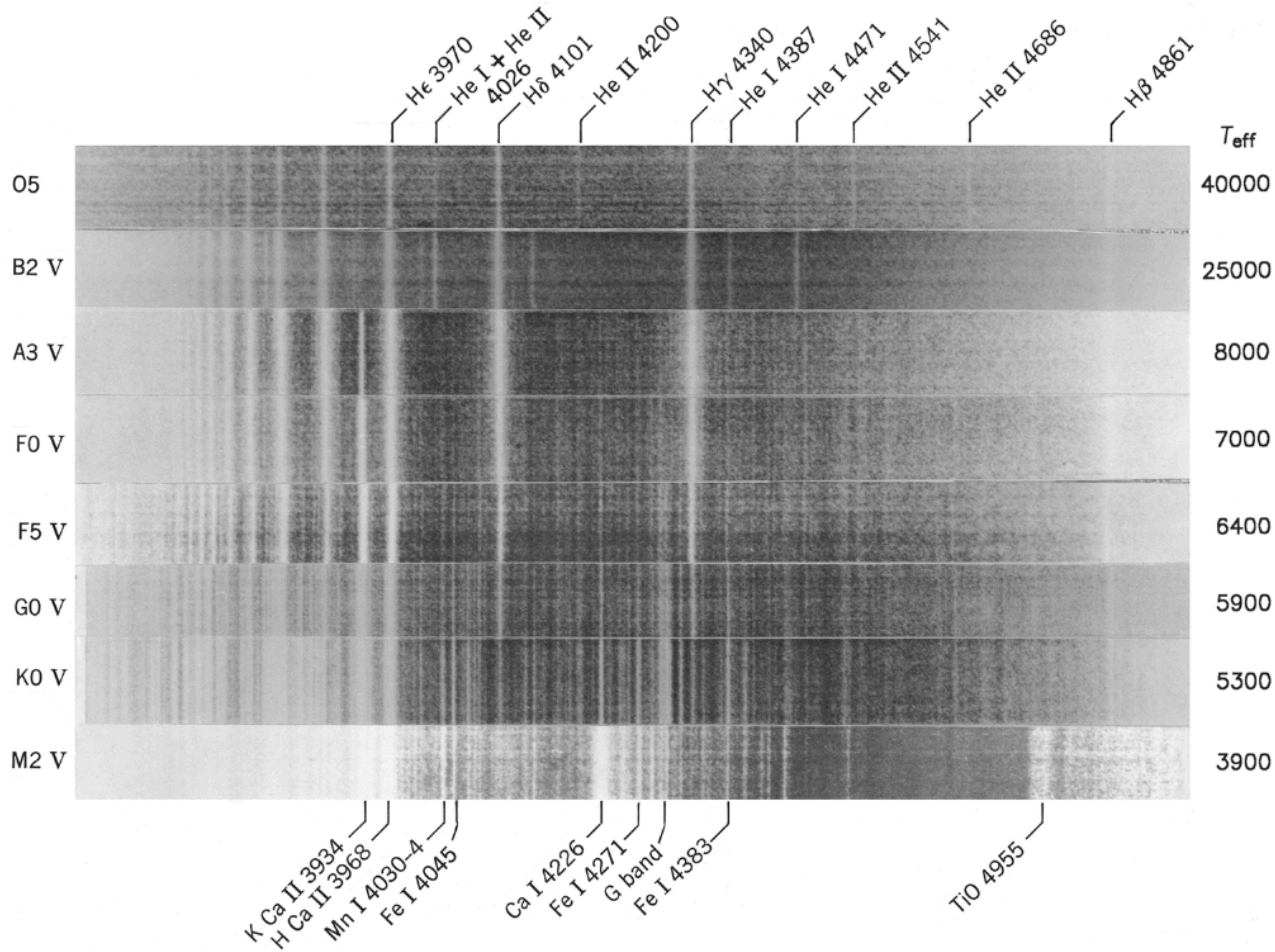
Surface gravity

$$g = g_\odot \frac{\mathfrak{M}}{R^2} \qquad g_\odot = 2.740 \times 10^4 \,\mathrm{cm/s^2}$$

$$L = 4\pi R^2 \int_0^\infty \mathfrak{F}_\nu \,\mathrm{d}\nu = 4\pi R^2 \sigma T_{\mathrm{eff}}^4$$

Introduction to Stellar Atmospheres

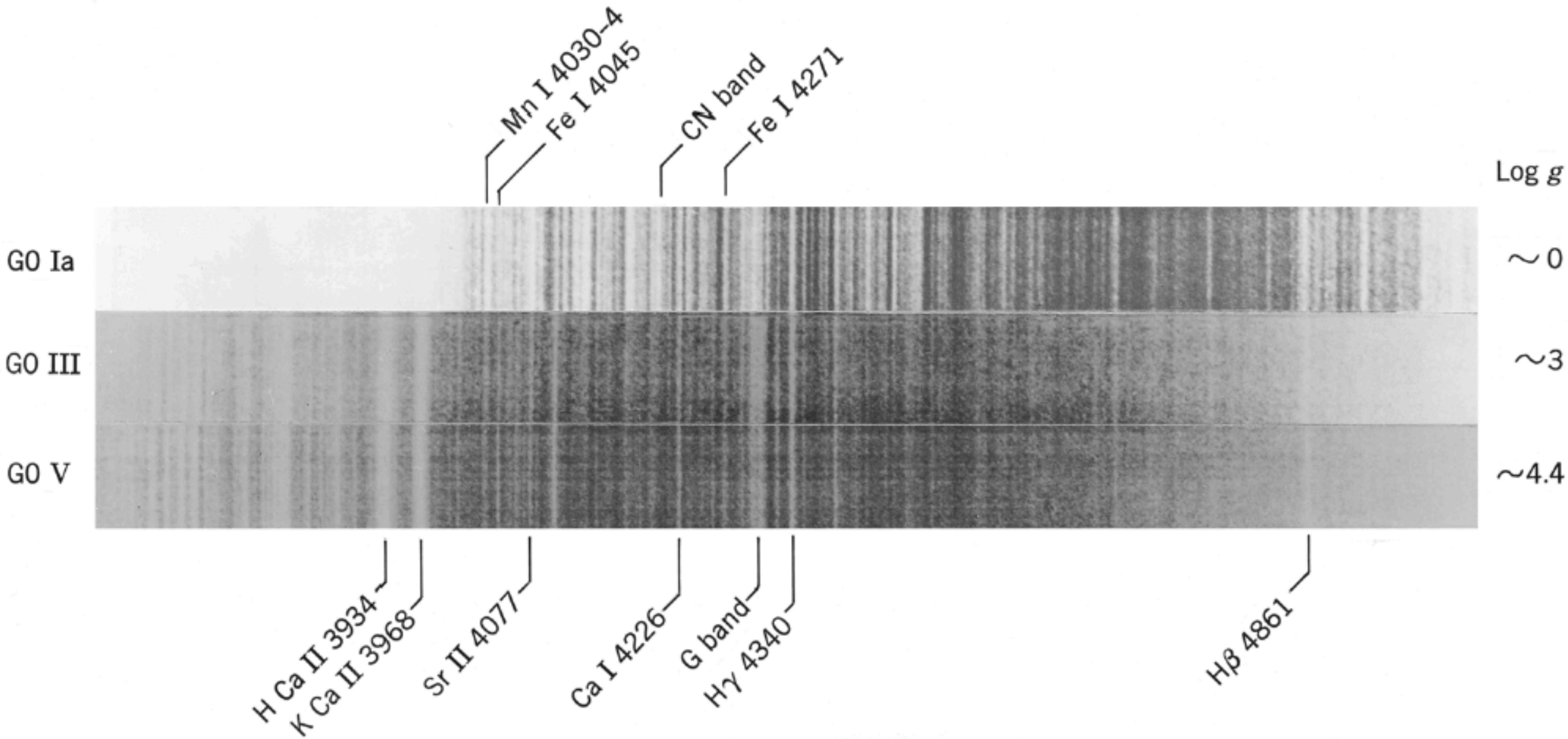
Spectral types



Introduction to Stellar Atmospheres

Luminosity classification

0, I, Ia, Ib, II, III, IV, and V



Introduction to Stellar Atmospheres

Spectral classification

Suffix notation

e	Emission lines are present
f	He II λ4686 and/or C III λ4650 in emission; mostly for O stars
k	Ca II K line when unexpected, e.g., interstellar in hot stars
m	Metallic; metal lines are stronger than normal
n	Nebulous; lines are broad and shallow; usually high rotation
nn	Very nebulous!
p	Peculiar; spectrum is abnormal
q	Queer; unusual emission; evolved from Q novae designation (archaic)
s	Sharp; lines are sharp, usually for early-type stars with low rotation
v	Variable; spectrum changes with time
w	Wolf–Rayet bands present (archaic)

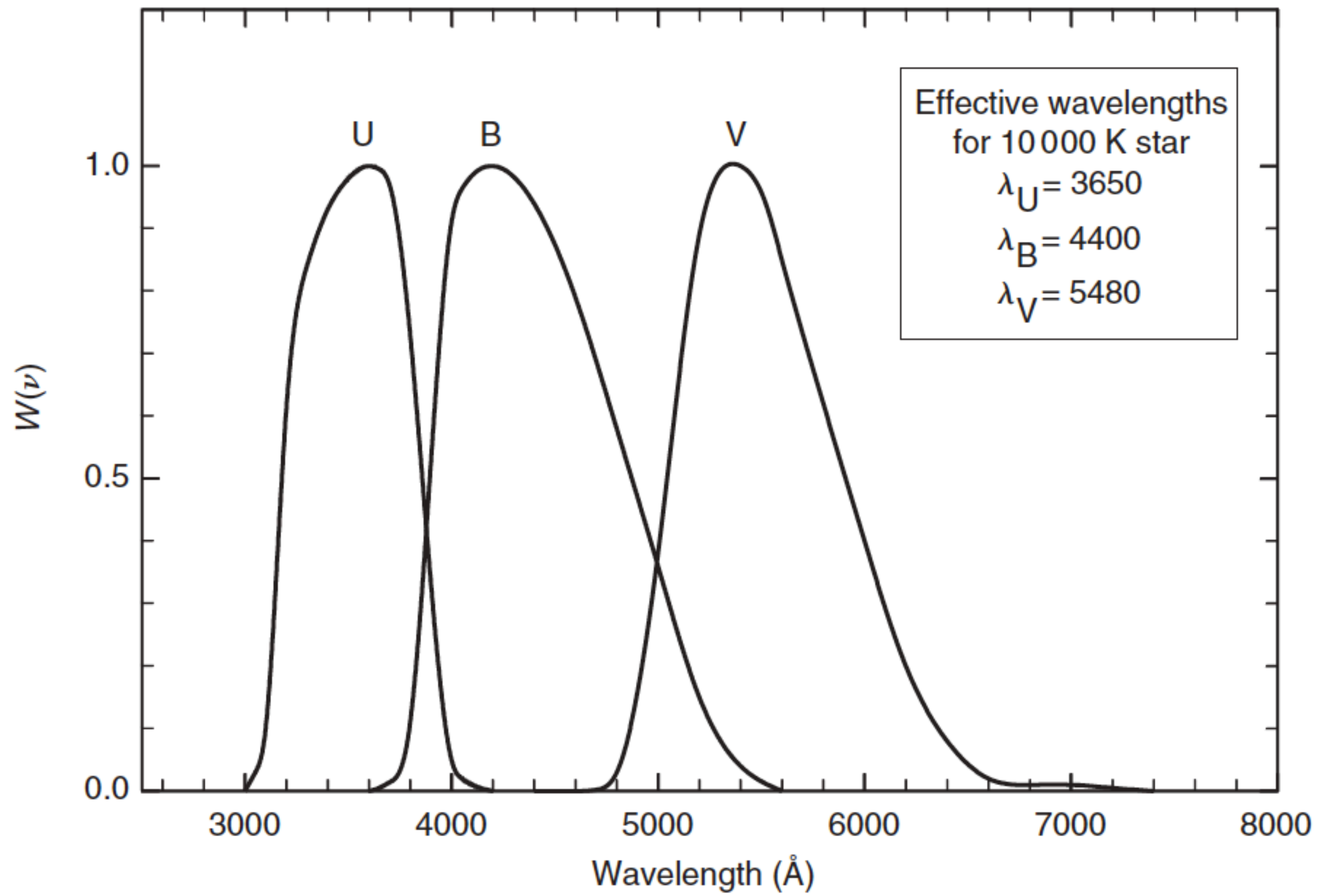
Old prefix notation

c for supergiants; g for giants; d for dwarfs

Introduction to Stellar Atmospheres

Magnitudes and color indices

$$m = -2.5 \log \int_0^\infty F_\nu W(\nu)\, \mathrm{d}\nu + \text{constant}$$

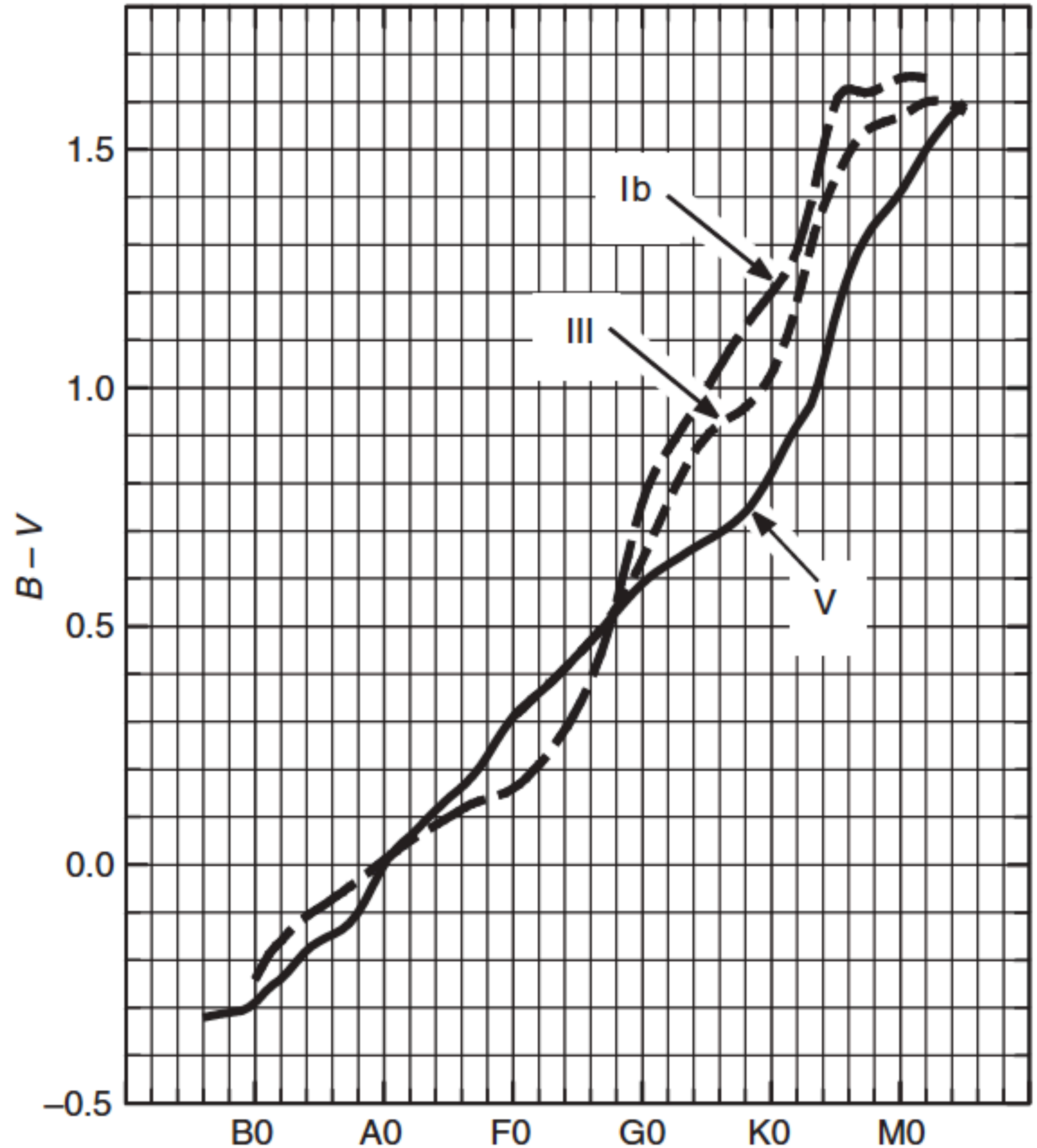


Introduction to Stellar Atmospheres

Magnitudes and color indices

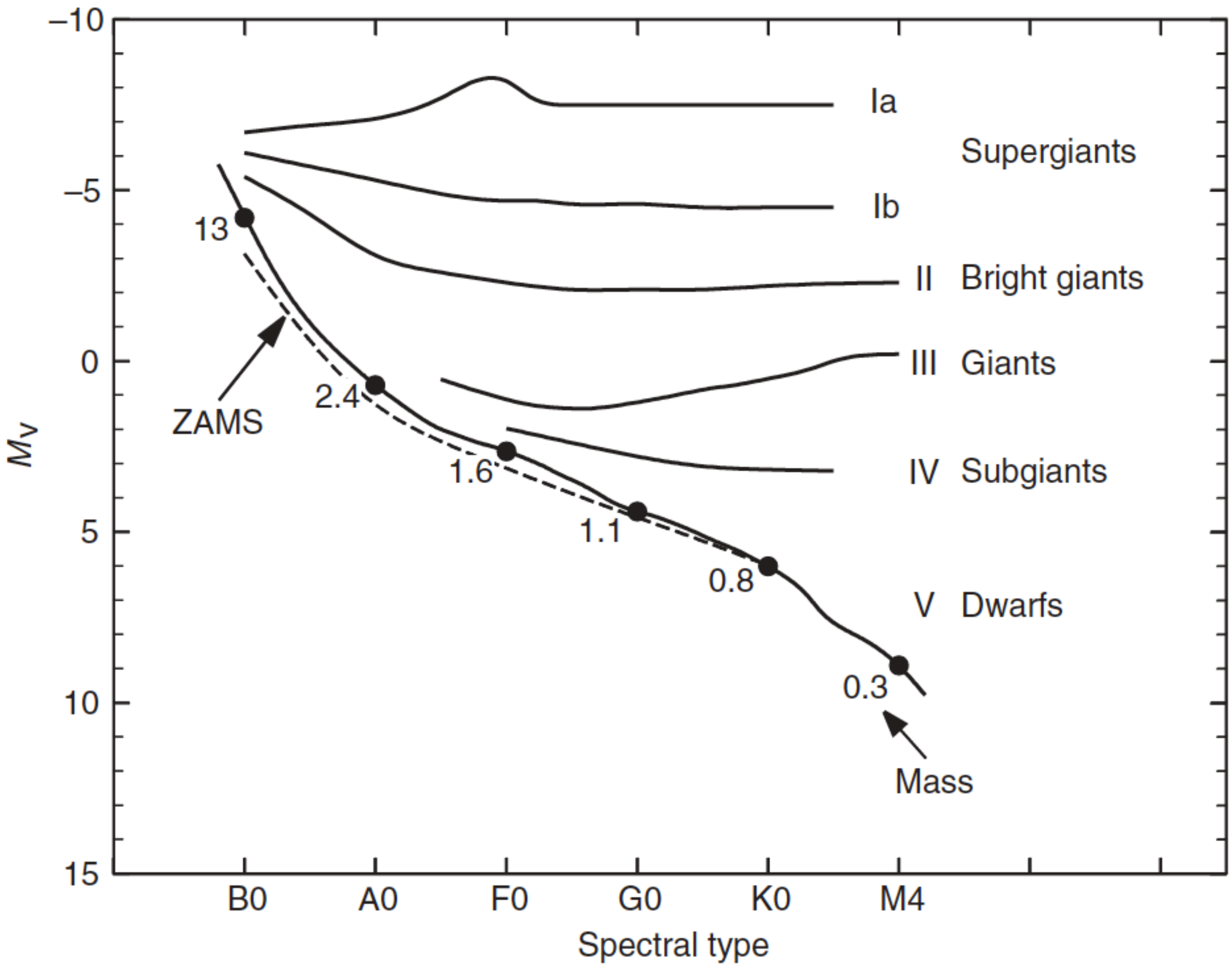
$$B - V = -2.5\log\left(\frac{\int F_\nu W_B(\nu)\,\mathrm{d}\nu}{\int F_\nu W_V(\nu)\,\mathrm{d}\nu}\right) + 0.710$$

$$U - B = -2.5\log\left(\frac{\int F_\nu W_U(\nu)\,\mathrm{d}\nu}{\int F_\nu W_B(\nu)\,\mathrm{d}\nu}\right) - 1.093$$



Introduction to Stellar Atmospheres

The Hertzsprung-Russell diagram



Plane-parallel Radiative Transfer

Formal Solutions

General Transport Equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial s} = \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = j_\nu - \alpha_\nu I_\nu$$

$$\frac{\partial I_\nu}{\partial s} = \frac{\partial I_\nu}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}$$

Spherical geometry

$$\frac{\partial I_\nu}{\partial s} = \cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} = \mu\,\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} = j_\nu - \alpha_\nu I_\nu$$

$$\frac{\mu}{\kappa_\nu \rho}\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{\kappa_\nu \rho r}\frac{\partial I_\nu}{\partial \mu} = S_\nu - I_\nu$$

Plane-parallel geometry

$$\mu\,\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu$$

Plane-parallel Radiative Transfer

Formal Solutions

Exponential integrals

$$S_\nu(\tau_\nu)\exp(-\tau_\nu) \to 0 \quad \tau_\nu \to \infty$$

$$I_\nu^-(0,\mu) = 0$$

$$\begin{aligned} I_\nu^+(\tau_\nu,\mu) &= +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu \\ I_\nu^-(\tau_\nu,\mu) &= +\int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/|\mu| \end{aligned}$$

$$\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\,\mu^n\,\mathrm{d}\mu$$

$$= \int_0^{+1}\mu^n\,\mathrm{d}\mu \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\frac{\mathrm{d}t_\nu}{\mu} + \int_{-1}^{0}\mu^n\,\mathrm{d}\mu \int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(\tau_\nu-t_\nu)/-\mu}\,\frac{\mathrm{d}t_\nu}{-\mu}$$

Plane-parallel Radiative Transfer

Formal Solutions

Exponential integrals

$$\int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\, \mu^n \, \mathrm{d}\mu$$

$$= \int_0^{+1} \mu^n \, \mathrm{d}\mu \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu} \frac{\mathrm{d}t_\nu}{\mu} + \int_{-1}^{0} \mu^n \, \mathrm{d}\mu \int_0^{\tau_\nu} S_\nu(t_\nu)\, \mathrm{e}^{-(\tau_\nu - t_\nu)/-\mu} \frac{\mathrm{d}t_\nu}{-\mu}$$

$$1/\mu = w \quad |\mathrm{d}\mu/\mathrm{d}w| = \mu/w.$$

$$\int_{-1}^{+1} I_\nu(\tau_\nu, \mu) \mu^n \, \mathrm{d}\mu$$

$$= \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, \mathrm{d}t_\nu \int_{\infty}^{1} \frac{\mathrm{e}^{-(t_\nu - \tau_\nu)\, w}}{w^{n+1}} \mathrm{d}w + (-1)^n \int_0^{\tau_\nu} S_\nu(t_\nu)\, \mathrm{d}t_\nu \int_{+1}^{\infty} \frac{\mathrm{e}^{-(\tau_\nu - t_\nu)\, w}}{w^{n+1}} \mathrm{d}w$$

$$= \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, E_{n+1}(t_\nu - \tau_\nu)\, \mathrm{d}t_\nu + (-1)^n \int_0^{\tau_\nu} S_\nu(t_\nu) E_{n+1}(\tau_\nu - t_\nu)\, \mathrm{d}t_\nu,$$

$$E_n(x) \equiv \int_1^{\infty} \frac{\mathrm{e}^{-xw}}{w^n} \mathrm{d}w = \int_0^1 \mathrm{e}^{-x/\mu}\, \mu^{n-1} \frac{\mathrm{d}\mu}{\mu}$$

Plane-parallel Radiative Transfer

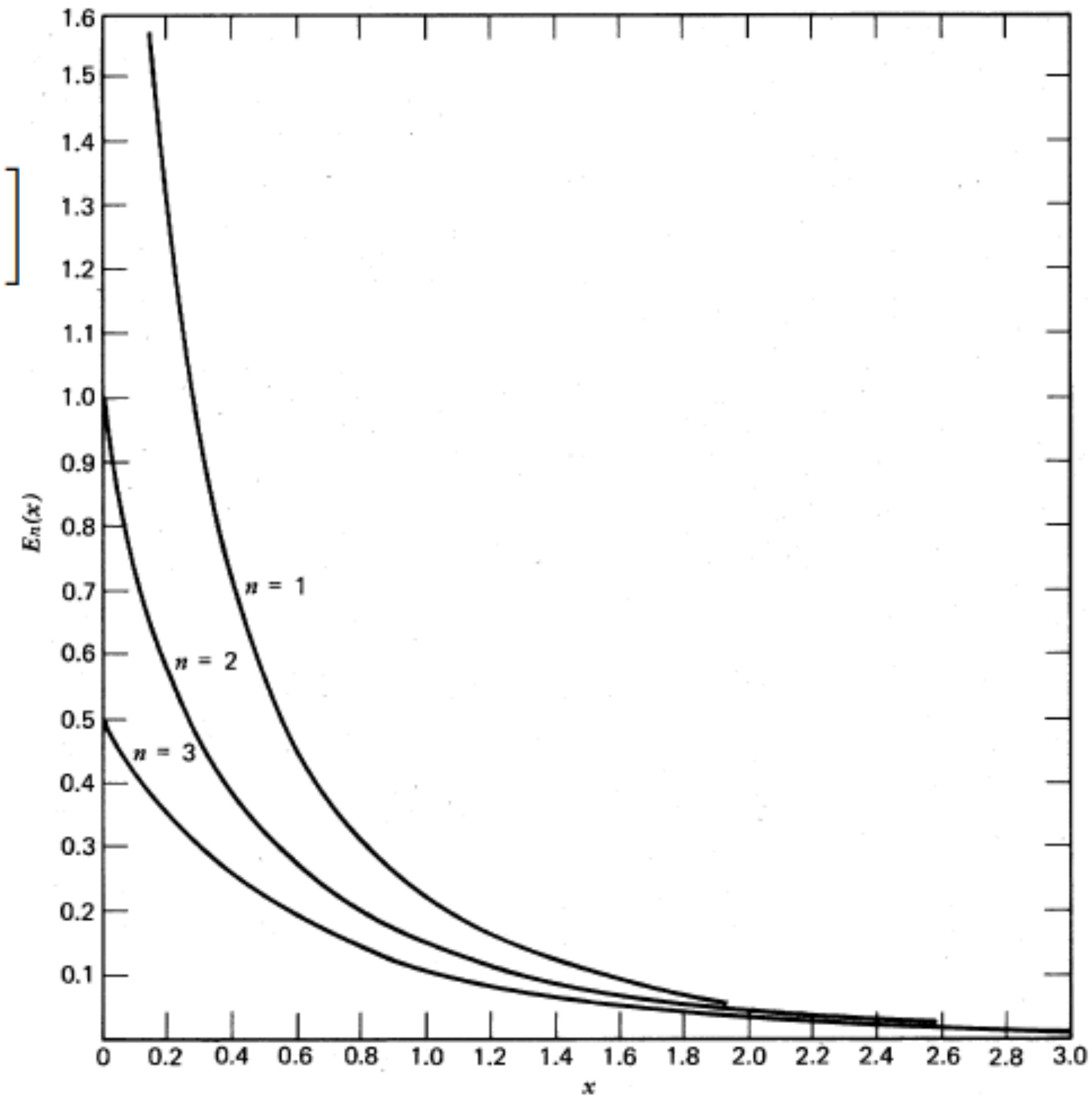
Formal Solutions

Exponential integrals

$$x \gg 1$$

$$E_n(x) = \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} + \cdots\right]$$

$$E_n \approx (1/x)\exp(-x)$$



Plane-parallel Radiative Transfer

Formal Solutions

Schwarzschild-Milne equations

$$
\begin{aligned}
J_\nu(\tau_\nu) &\equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\,\mathrm{d}\mu \\
&= \frac{1}{2}\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,E_1(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu + \frac{1}{2}\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_1(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu \\
&= \frac{1}{2}\int_0^{\infty} S_\nu(t_\nu)\,E_1(|t_\nu-\tau_\nu|)\,\mathrm{d}t_\nu,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_\nu(\tau_\nu) &= \mathcal{F}_\nu^+(\tau_\nu) - \mathcal{F}_\nu^-(\tau_\nu) \\
&= 2\pi\int_0^1 \mu I_\nu(\tau_\nu)\,\mathrm{d}\mu \;-\; 2\pi\int_0^{-1} \mu I_\nu(\tau_\nu)\,\mathrm{d}\mu \\
&= 2\pi\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu - 2\pi\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu
\end{aligned}
$$

$$
K_\nu(\tau_\nu) = \frac{1}{2}\int_0^{\infty} S_\nu(t_\nu)\,E_3(|t_\nu-\tau_\nu|)\,\mathrm{d}t_\nu
$$

Plane-parallel Radiative Transfer

Formal Solutions

Schwarzschild-Milne equations

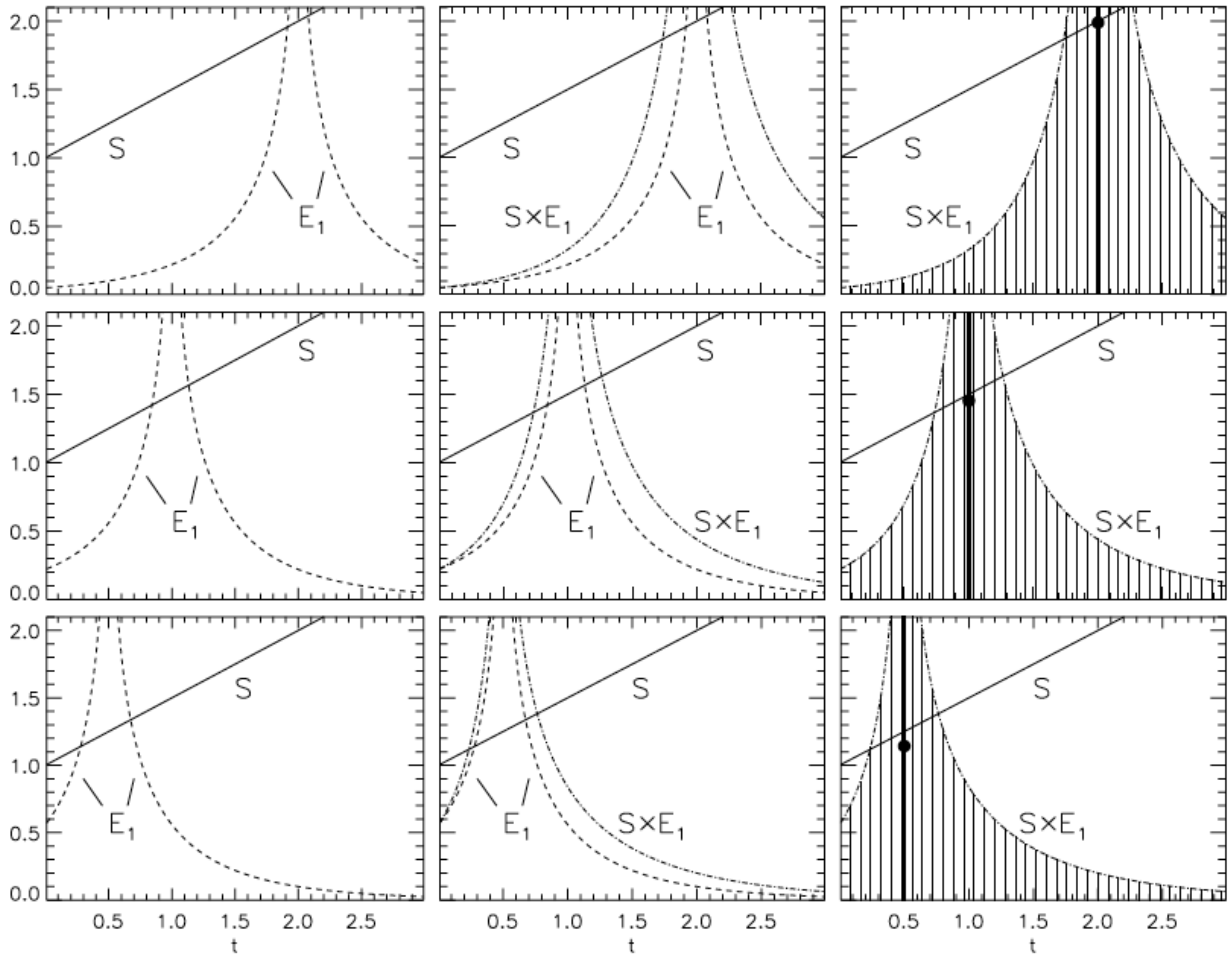
Surface values

$$
\begin{aligned}
I_\nu^+(0,\mu) &= \int_0^\infty S_\nu(\tau_\nu)\, \mathrm{e}^{-\tau_\nu/\mu}\, \mathrm{d}\tau_\nu/\mu \\
\mathcal{F}_\nu^+(0) &= 2\pi \int_0^\infty S_\nu(\tau_\nu)\, E_2(\tau_\nu)\, \mathrm{d}\tau_\nu
\end{aligned}
$$

Plane-parallel Radiative Transfer

Formal Solutions

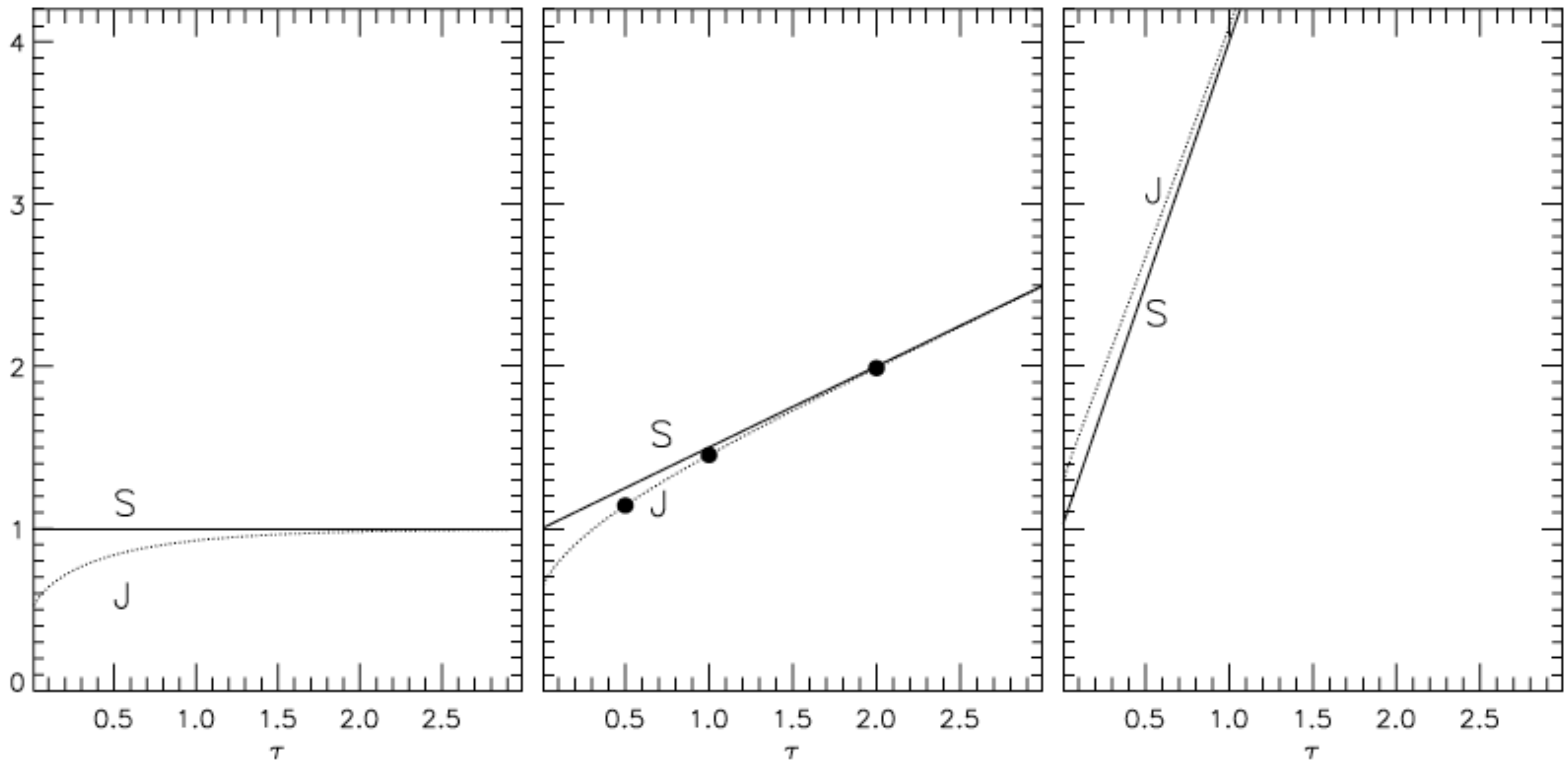
Schwarzschild-Milne equations



Plane-parallel Radiative Transfer

Formal Solutions

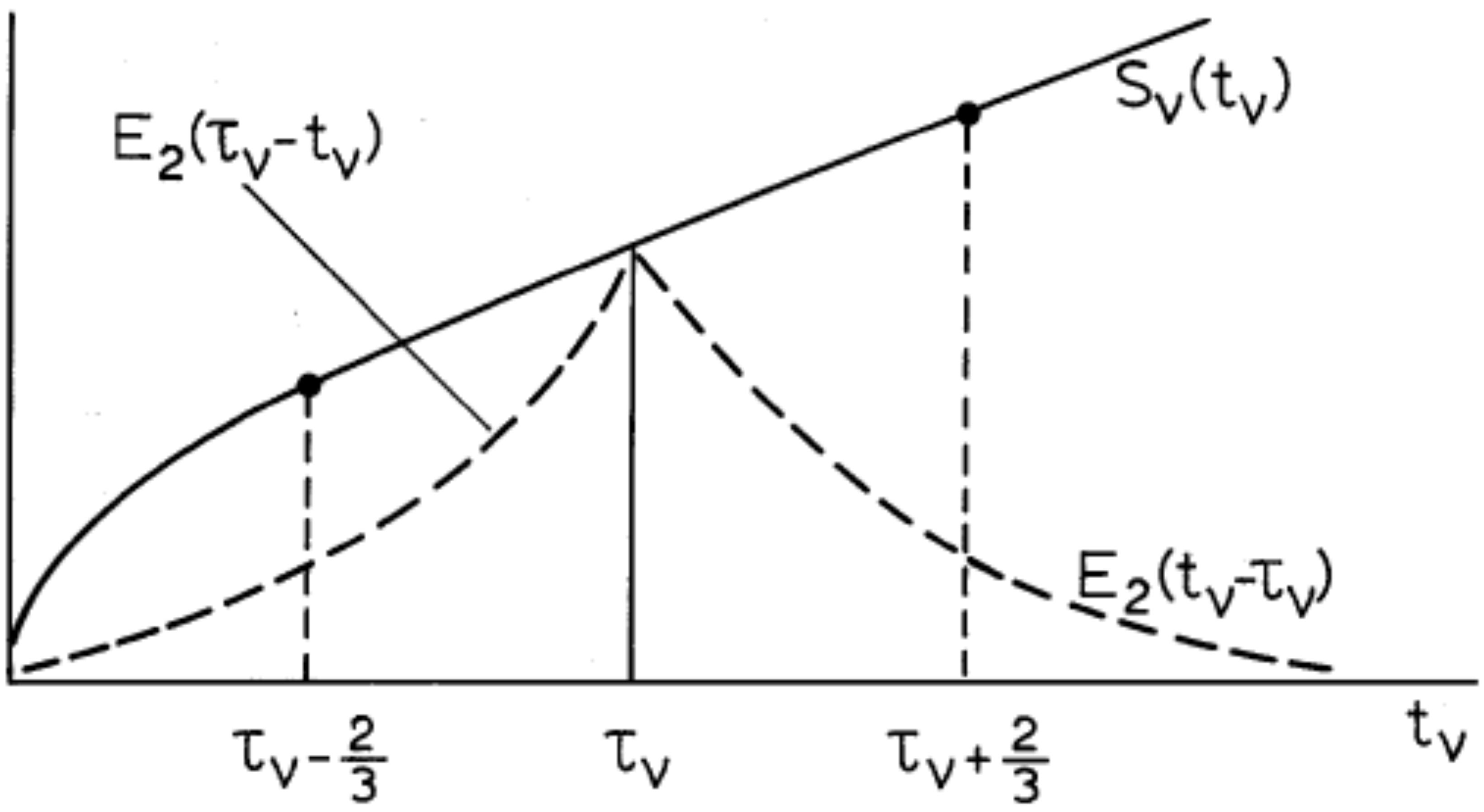
Schwarzschild-Milne equations



Plane-parallel Radiative Transfer

Formal Solutions

Schwarzschild-Milne equations



Plane-parallel Radiative Transfer

Formal Solutions

Operators

Laplace transform

$$\mathcal{L}_{1/\mu}[S_\nu(\tau_\nu)] \equiv \int_0^\infty S_\nu(\tau_\nu)\, \mathrm{e}^{-\tau_\nu/\mu}\, \mathrm{d}\tau_\nu/\mu = I_\nu^+(0,\mu)$$

Classical Lambda operator

$$\mathbf{\Lambda}_\tau[f(t)] \equiv \frac{1}{2}\int_0^\infty f(t)\, E_1(|t-\tau|)\, \mathrm{d}t$$

$$\begin{aligned}
\mathbf{\Lambda}_\tau[1] &= 1 - \frac{1}{2}E_2(\tau) \\
\mathbf{\Lambda}_\tau[t] &= \tau + \frac{1}{2}E_3(\tau) \\
\mathbf{\Lambda}_\tau[t^2] &= \frac{2}{3} + \tau^2 - E_4(\tau) \\
\mathbf{\Lambda}_\tau[t^p] &= \frac{1}{2}p!\left[\sum_{k=0}^{p}\frac{\tau^k}{k!}\delta_\alpha + (-1)^{p+1}E_{p+2}(\tau)\right]
\end{aligned}$$

$\delta_\alpha = 0$ for even $\alpha \equiv p+1-k$ and $\delta_\alpha = 2/\alpha$ for odd α

Plane-parallel Radiative Transfer

Formal Solutions

Operators

Classical Lambda operator

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu - \tau_\nu|)\, \mathrm{d}t_\nu = \mathbf{\Lambda}_{\tau_\nu}[S_\nu(t_\nu)]$$

Phi and Chi operator

$$\begin{aligned}\mathbf{\Phi}_{\tau_\nu}[S_\nu(t_\nu)] &\equiv 2\int_{\tau_\nu}^\infty S_\nu(t_\nu)\, E_2(t_\nu - \tau_\nu)\, \mathrm{d}t_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)\, E_2(\tau_\nu - t_\nu)\, \mathrm{d}t_\nu \\ &= F_\nu(\tau_\nu)\end{aligned}$$

$$F_\nu = \mathcal{F}_\nu/\pi$$

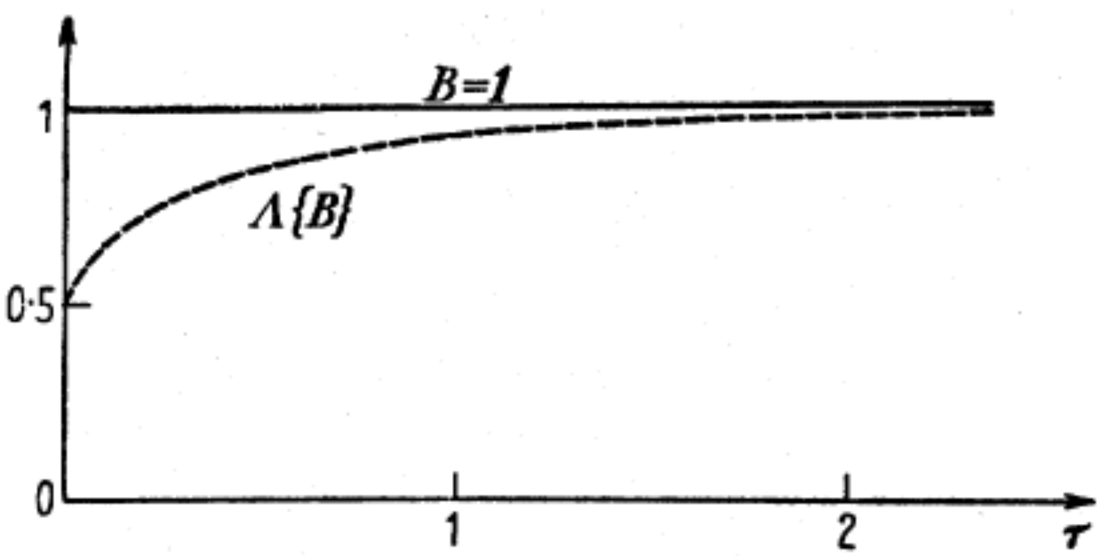
$$\begin{aligned}\chi_{\tau_\nu}[S_\nu(t_\nu)] &\equiv 2\int_0^\infty S_\nu(t_\nu)\, E_3(|t_\nu - \tau_\nu|)\, \mathrm{d}t_\nu \\ &= 4\, K_\nu(\tau_\nu),\end{aligned}$$

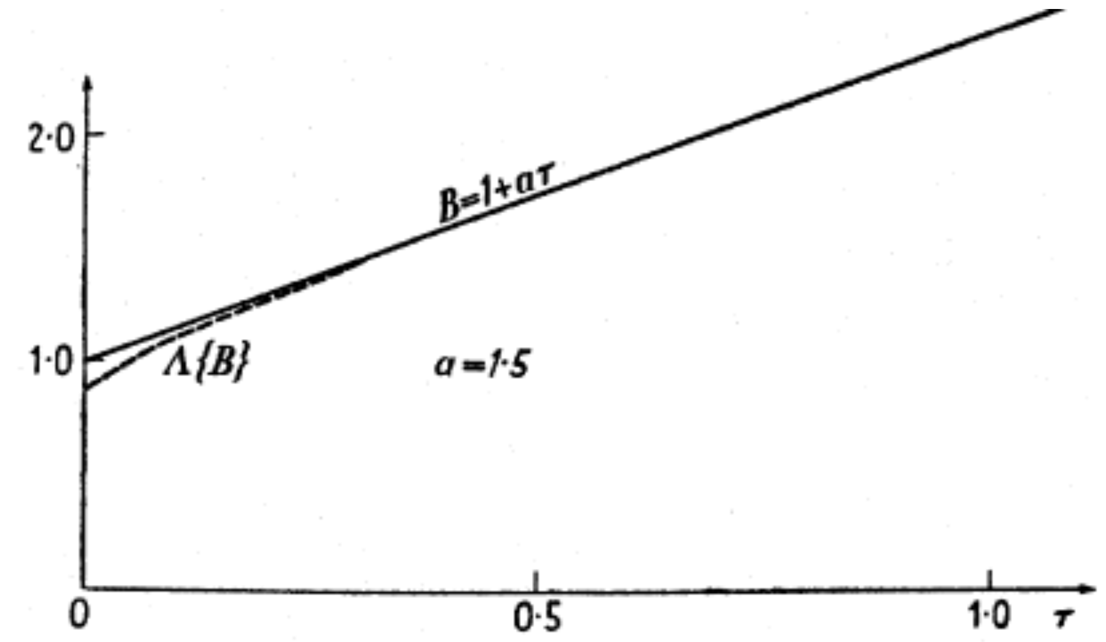
$$\mathbf{\Phi}_\tau[f(t)] = \frac{\mathrm{d}}{\mathrm{d}\tau}\chi_\tau[f(t)]$$

Plane-parallel Radiative Transfer

Formal Solutions

Classical Lambda operator

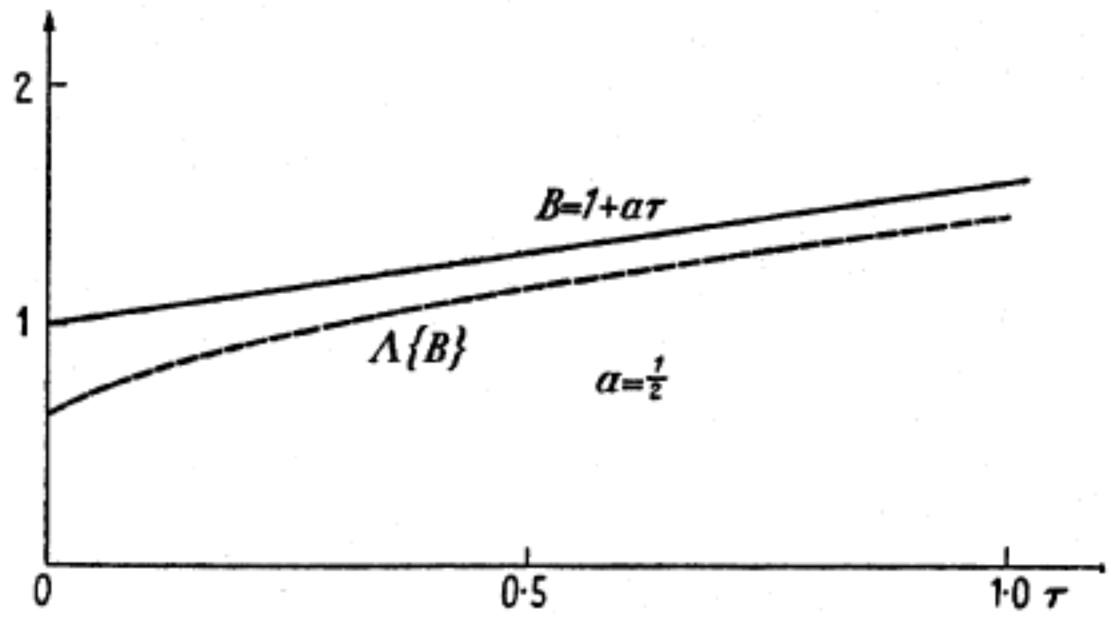


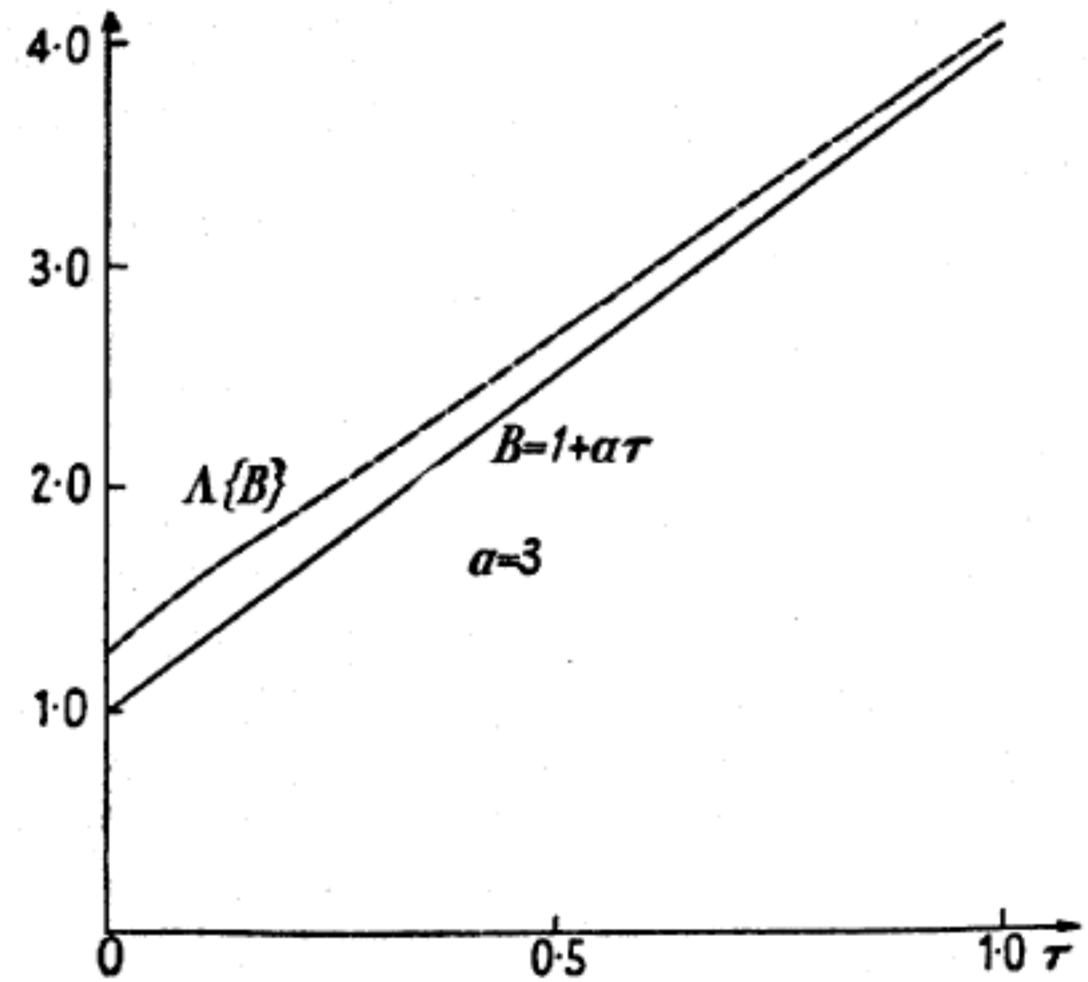


Plane-parallel Radiative Transfer

Formal Solutions

Classical Lambda operator

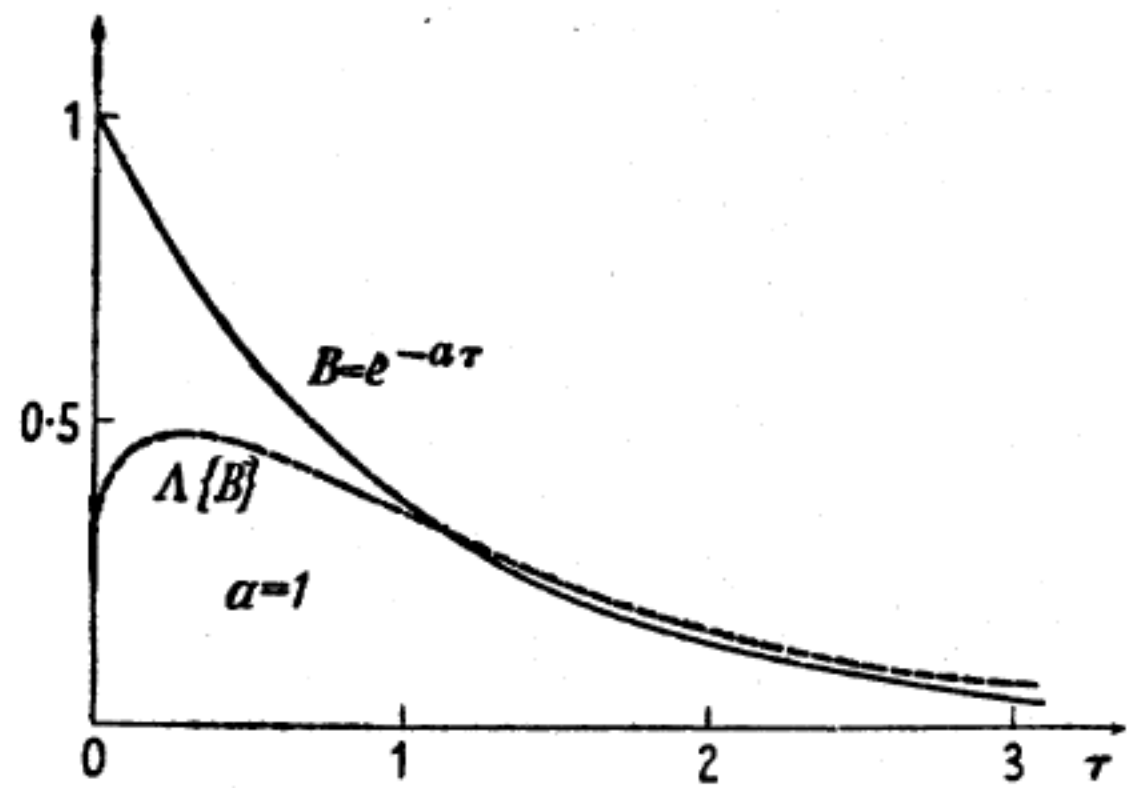


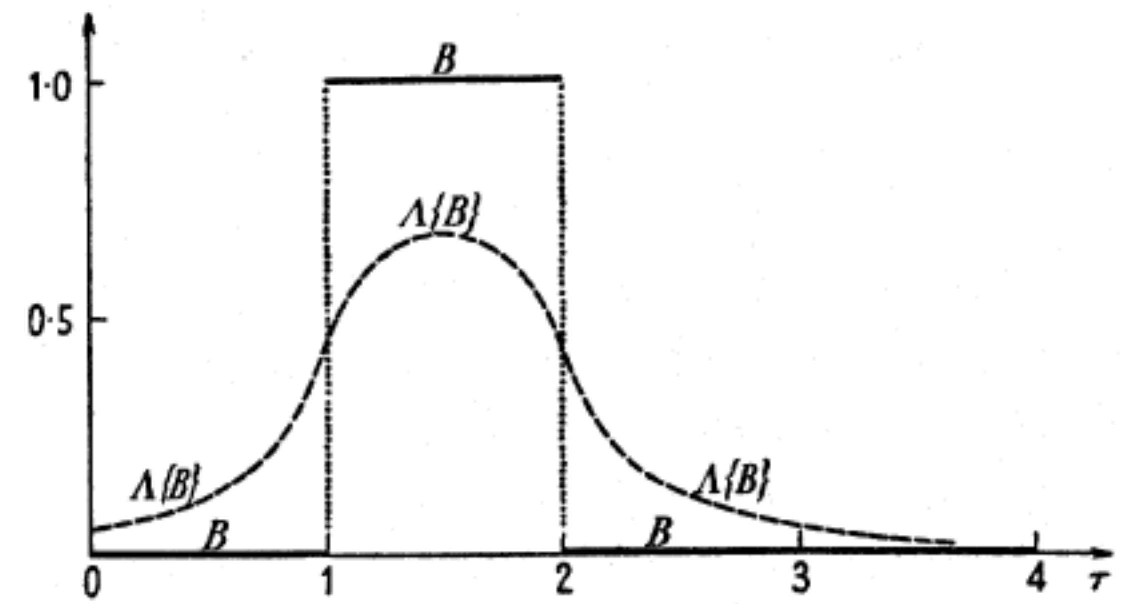


Plane-parallel Radiative Transfer

Formal Solutions

Classical Lambda operator

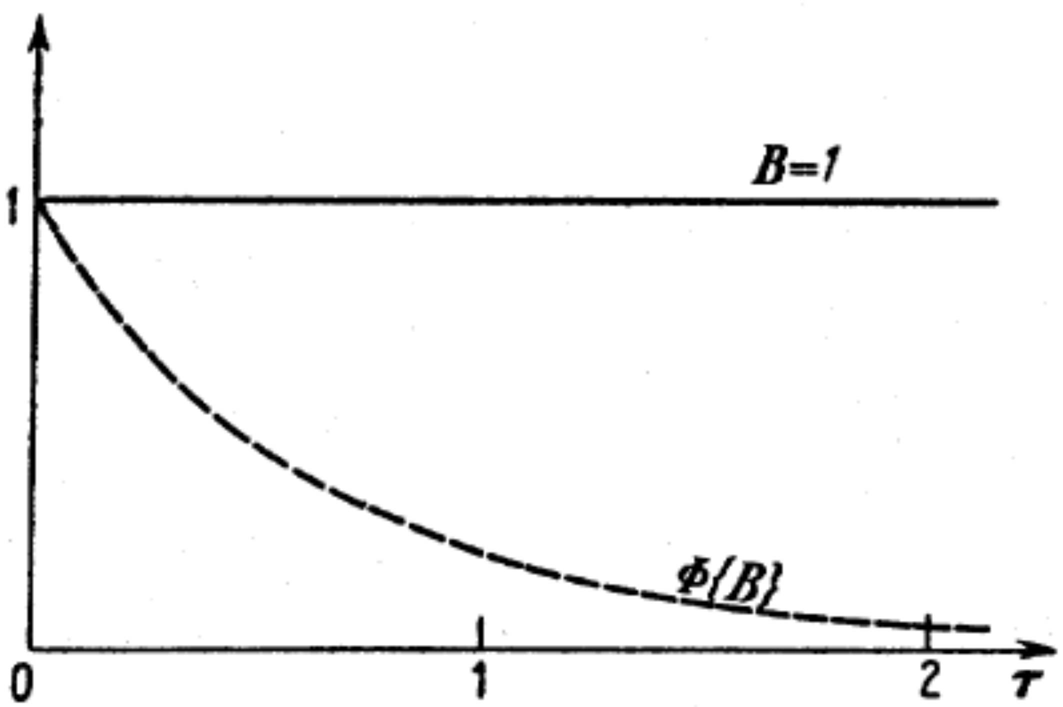


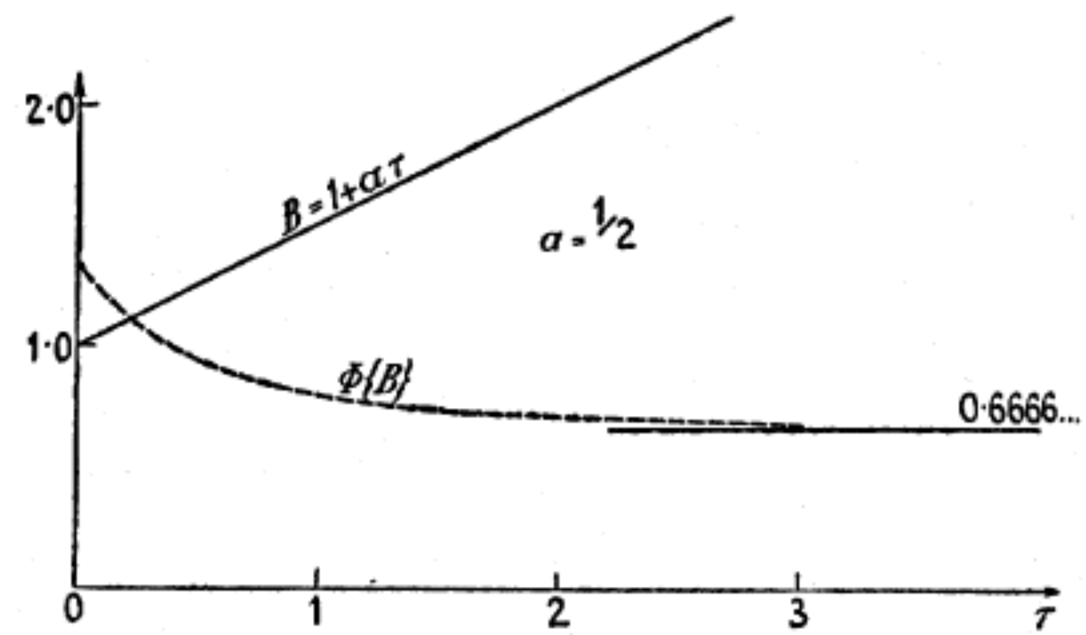


Plane-parallel Radiative Transfer

Formal Solutions

Classical Lambda operator

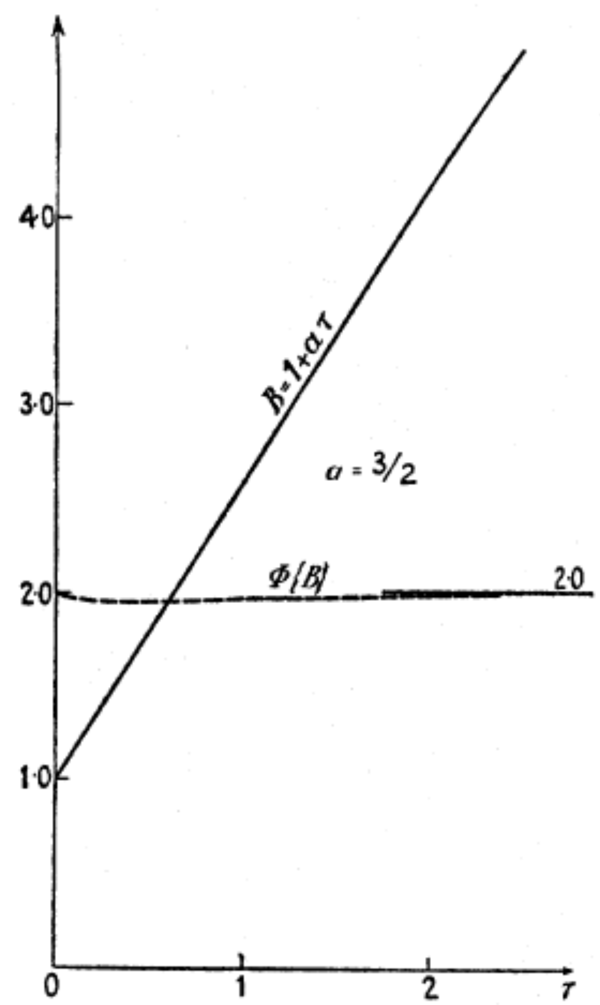


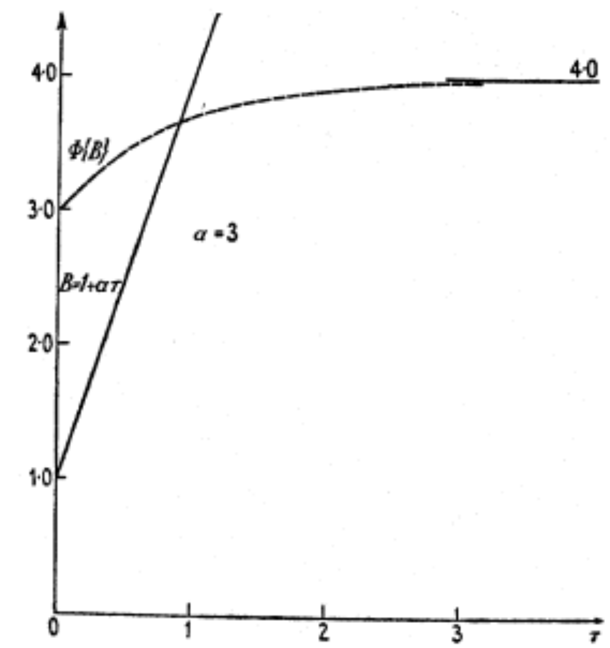


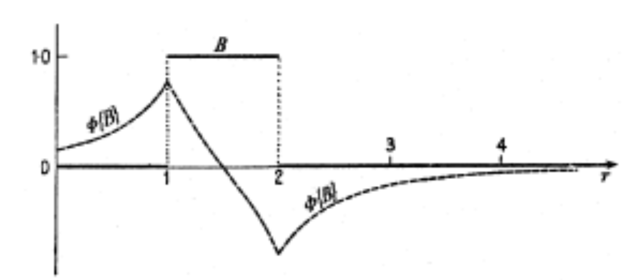
Plane-parallel Radiative Transfer

Formal Solutions

Classical Lambda operator







Plane-parallel Radiative Transfer

Formal Solutions

Generalised Lambda operators

$$I_\nu(\tau_\nu, \mu) = \mathbf{\Lambda}_{\mu\nu}[S_\nu(t_\nu)]$$

$$J_\nu(\tau_\nu) = \mathbf{\Lambda}_\nu[S_\nu(t_\nu)],$$

$$\mathbf{\Lambda}_\nu = \frac{1}{2}\int_{-1}^{+1} \mathbf{\Lambda}_{\mu\nu}\,\mathrm{d}\mu$$

$$\begin{aligned}\mathbf{\Lambda}_{+\mu\nu}[S_\nu] = I_\nu^+(\tau_\nu, \mu) &= \int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu \\ &= \mathrm{e}^{\tau_{\nu\mu}} \int_{\tau_{\nu\mu}}^{\infty} S_\nu\,\mathrm{e}^{-t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu}\end{aligned}$$

$$\begin{aligned}\mathbf{\Lambda}_{-\mu\nu}[S_\nu] = I_\nu^-(\tau_\nu, -|\mu|) &= \int_0^{\tau_\nu} S_\nu(t_\nu)\,\mathrm{e}^{-(\tau_\nu-t_\nu)/|\mu|}\,\mathrm{d}t_\nu/|\mu| \\ &= \mathrm{e}^{-\tau_{\nu\mu}} \int_0^{\tau_{\nu\mu}} S_\nu\,\mathrm{e}^{t_{\nu\mu}}\,\mathrm{d}t_{\nu\mu},\end{aligned}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at the surface

Eddington-Barbier approximations

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n {\tau_\nu}^n$$

$$\begin{aligned}
I_\nu^+(0,\mu) &= \mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} \\
&= \sum_{n=0}^{\infty} n!\, a_n \mu^n \\
J_\nu(\tau_\nu) &= \mathbf{\Lambda}_\nu[S_\nu] \\
&= a_0\mathbf{\Lambda}_\nu[1] + a_1\mathbf{\Lambda}_\nu[t] + a_2\mathbf{\Lambda}_\nu[t^2] + \cdots \\
&\approx a_0\left[1 - \frac{1}{2}E_2(\tau_\nu)\right] + a_1\left[\tau_\nu + \frac{1}{2}E_3(\tau_\nu)\right] + a_2\left[\frac{2}{3} + \tau_\nu^2 - E_4(\tau_\nu)\right] \\
&\approx a_0 + a_1\tau_\nu + a_2\tau_\nu^2 + \frac{2}{3}a_2 - \frac{a_0}{2}E_2(\tau_\nu) + \frac{a_1}{2}E_3(\tau_\nu) - a_2E_4(\tau_\nu) \\
F_\nu(\tau_\nu) &= \mathbf{\Phi}_{\tau_\nu}[S_\nu] \\
&= 2a_0E_3(\tau_\nu) + a_1\left[\frac{4}{3} - 2E_4(\tau_\nu)\right] + a_2\left[\frac{8}{3}\tau_\nu + 4E_5(\tau_\nu)\right] + \cdots
\end{aligned}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at the surface

Eddington-Barbier approximations

$$
\begin{aligned}
I_\nu^+(0,\mu) &\approx a_0 + a_1\mu \\
&\approx S_\nu(\tau_\nu = \mu), \\
J_\nu(0) &\approx a_0 + \frac{2a_2}{3} - \frac{a_0}{2} + \frac{a_1}{4} - \frac{a_2}{3} \\
&\approx \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{3} \\
&\approx \frac{1}{2} S_\nu(\tau_\nu = 1/2), \\
F_\nu(0) &= a_0 + \frac{2}{3}a_1 + a_2 + \cdots \\
&\approx S_\nu(\tau_\nu = \frac{2}{3}),
\end{aligned}
$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at the surface

Second Eddington approximations

$$S_\nu = a_0$$

$$I_\nu^+(0,\mu) = S_\nu = a_0 \qquad \mu > 0$$

$$J_\nu(0) = S_\nu/2 = a_0/2 = I_\nu(0)/2$$

$$F_\nu(0) = S_\nu = I_\nu(0) = a_0$$

$$F_\nu(0) = 2J_\nu(0) = 4H_\nu(0)$$

$$\mathcal{F}_\nu(0) = 2\pi J_\nu(0)$$

$$\begin{aligned} F_\nu(0) &\equiv 2\int_{-1}^{+1} I_\nu(0,\mu)\mu \,\mathrm{d}\mu \\ &= 2\int_0^1 I_\nu(0,\mu)\mu \,\mathrm{d}\mu \\ &\approx 2 < I_\nu^+(0,\mu) > \int_0^1 \mu \,\mathrm{d}\mu \\ &\approx 2\, J_\nu(0), \end{aligned}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Taylor expansion

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}$$

$$I_\nu^+(\tau_\nu, \mu) \;=\; +\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu}\, \mathrm{d}t_\nu/\mu$$

$$\int_{x_1}^{x_2} x^n\, \mathrm{e}^{ax}\, \mathrm{d}x \;=\; \frac{x^n\, \mathrm{e}^{ax}}{a}\Big/_{x_1}^{x_2} - \frac{n}{a} \int_{x_1}^{x_2} x^{n-1}\, \mathrm{e}^{ax}\, \mathrm{d}x$$

$$\int_{x_1}^{x_2} x^n\, \mathrm{e}^{-x}\, \mathrm{d}x \;=\; -\,\mathrm{e}^{-x} \left[x^n + n x^{n-1} + n(n-1) x^{n-2} + \cdots + n!\right]\Big/_{x_1}^{x_2}$$

$$\int_0^{\infty} x^n\, \mathrm{e}^{-x}\, \mathrm{d}x \;=\; n!$$

$$I_\nu^+(\tau_\nu, \mu) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \int_{\tau_\nu}^{\infty} (t_\nu - \tau_\nu)^n\, \mathrm{e}^{-(t_\nu - \tau_\nu)/\mu}\, \mathrm{d}t_\nu/\mu$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Taylor expansion

$$
\begin{aligned}
I_\nu^+(\tau_\nu,\mu) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \int_{\tau_\nu}^{\infty} (t_\nu - \tau_\nu)^n \, \mathrm{e}^{-(t_\nu-\tau_\nu)/\mu} \, \mathrm{d}t_\nu/\mu \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \frac{1}{n!} \int_0^{\infty} x^n \, \mathrm{e}^{-x/\mu} \, \mathrm{d}x/\mu \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \frac{\mu^n}{n!} \int_0^{\infty} x^n \, \mathrm{e}^{-x} \, \mathrm{d}x \\
&= \sum_{n=0}^{\infty} \mu^n \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}
\end{aligned}
$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Taylor expansion

$$
\begin{aligned}
I_\nu^-(\tau_\nu,\mu) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \left[-\int_0^{\tau_\nu} (t_\nu-\tau_\nu)^n \, \mathrm{e}^{-(t_\nu-\tau_\nu)/\mu} \, \mathrm{d}t_\nu/\mu\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} (-1)^n \left[+\int_0^{\tau_\nu} (\tau_\nu-t_\nu)^n \, \mathrm{e}^{-(\tau_\nu-t_\nu)/|\mu|} \, \mathrm{d}t_\nu/|\mu|\right] \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} (-1)^n \frac{|\mu|^n}{n!} \left[\int_0^{\tau_\nu/|\mu|} x^n \, \mathrm{e}^{-x} \, \mathrm{d}x\right] \\
&= \sum_{n=0}^{\infty} \mu^n \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \left[1 - \frac{\mathrm{e}^{-(\tau_\nu/|\mu|)}}{n!}\left\{(\tau_\nu/|\mu|)^n + n(\tau_\nu/|\mu|)^{n-1} + \cdots + n!\right\}\right]
\end{aligned}
$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Taylor expansion

$$I_\nu(\tau_\nu,\mu) = S_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} + \mu^2 \left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots$$

Large depth

$$\tau_\nu \gg 1$$

$$J_\nu(\tau_\nu) = \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}\int_{-1}^{+1}\mu^n\,\mathrm{d}\mu = \sum_{k=0}^{\infty}\frac{1}{2k+1}\left[\frac{\mathrm{d}^{(2k)} S_\nu(t_\nu)}{\mathrm{d}t_\nu^{(2k)}}\right]_{\tau_\nu}$$

$$J_\nu(\tau_\nu) = S_\nu(\tau_\nu) + \frac{1}{3}\left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots$$

$$F_\nu(\tau_\nu) = \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} + \frac{4}{5}\left[\frac{\mathrm{d}^3 S_\nu(t_\nu)}{\mathrm{d}t_\nu^3}\right]_{\tau_\nu} + \cdots$$

$$K_\nu(\tau_\nu) = \frac{1}{3}S_\nu(\tau_\nu) + \frac{1}{5}\left[\frac{\mathrm{d}^2 S_\nu(t_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu} + \cdots$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Convergence

$$\left|\frac{\mathrm{d}^n S_\nu}{\mathrm{d}t_\nu^n}\right| \sim \frac{S_\nu}{t_\nu^n}$$

$$\frac{|\mathrm{d}^{n+2} S_\nu/\mathrm{d}t_\nu^{n+2}|}{|\mathrm{d}^n S_\nu/\mathrm{d}t_\nu^n|} \sim \frac{S_\nu/t_\nu^{n+2}}{S_\nu/t_\nu^n} \sim \frac{1}{t_\nu^2}$$

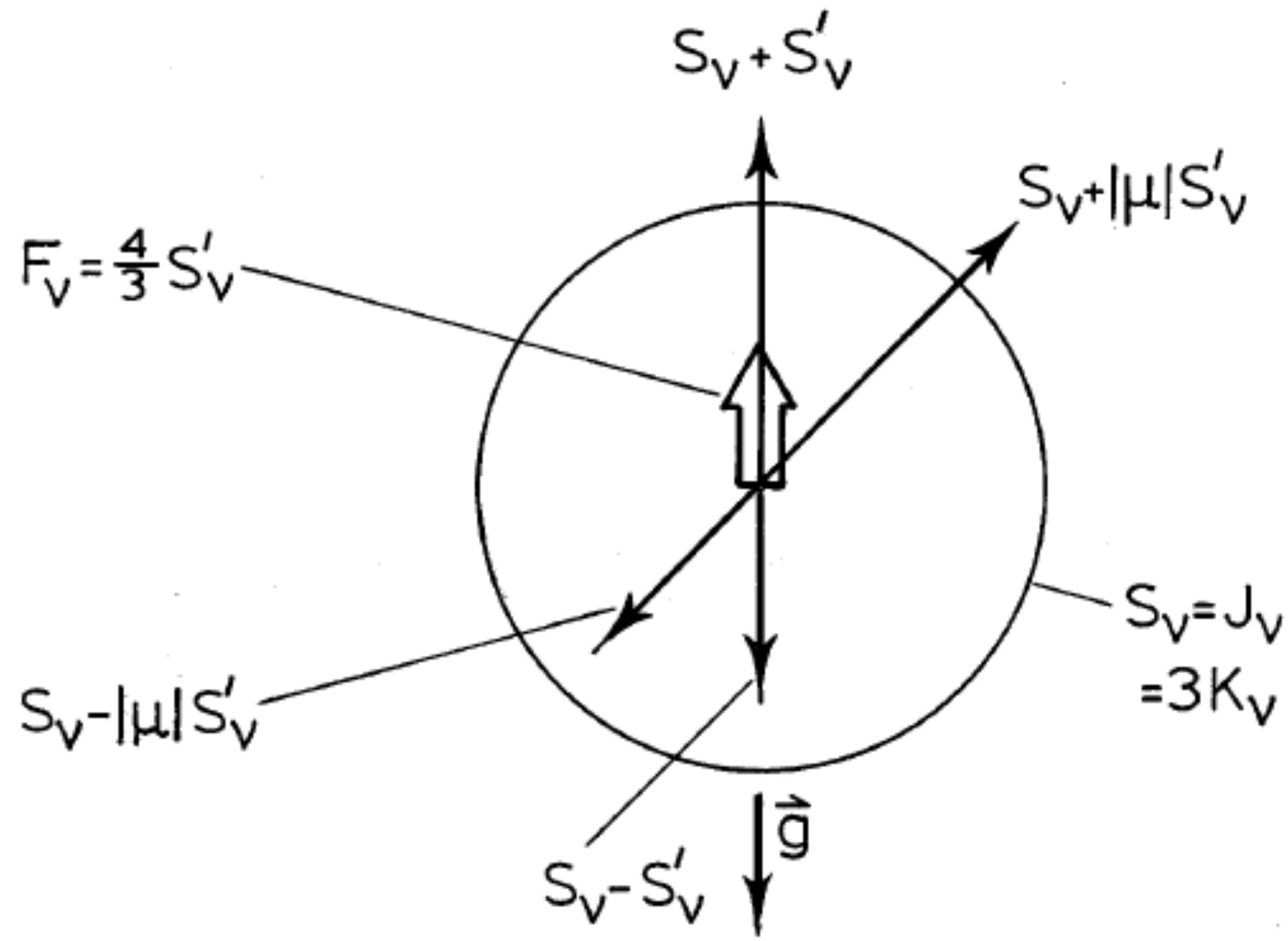
$$\begin{aligned}
I_\nu(\tau_\nu, \mu) &\approx S_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \\
J_\nu(\tau_\nu) &\approx S_\nu(\tau_\nu) \\
F_\nu(\tau_\nu) &\approx \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \\
K_\nu(\tau_\nu) &\approx \frac{1}{3} S_\nu(\tau_\nu).
\end{aligned}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

$$\frac{\mathrm{d}S_\nu/\mathrm{d}\tau_\nu}{S_\nu} \sim \frac{S_\nu/\tau_\nu}{S_\nu} \sim \frac{1}{\tau_\nu}$$



Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Diffusion approximation

$$\tau_\nu^* > 1$$

$$S_\nu = B_\nu$$

$$\begin{aligned}
I_\nu(\tau_\nu, \mu) &\approx B_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}B_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \\
J_\nu(z) &\approx B_\nu(z) \\
\mathcal{F}_\nu(z) &\approx 2\pi \int_{-1}^{+1} \mu\, I_\nu\, \mathrm{d}\mu \approx \frac{4\pi}{3} \frac{\mathrm{d}B_\nu(z)}{\mathrm{d}\tau_\nu}
\end{aligned}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Rosseland mean extinction

$$\frac{1}{\alpha_{\rm R}} \equiv \frac{\int_0^\infty (1/\alpha_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}$$

$$\frac{1}{\kappa_{\rm R}} \equiv \frac{\int_0^\infty (1/\kappa_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}$$

$$\kappa_{\rm R}(z) = \alpha_{\rm R}(z)/\rho(z)$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Total radiative energy diffusion

$$
\begin{aligned}
\mathcal{F}(z) &\equiv \int_0^\infty \mathcal{F}_\nu(z)\,\mathrm{d}\nu \\
&\approx -\frac{4\pi}{3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}z}\,\mathrm{d}\nu \\
&\approx -\frac{4\pi}{3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}z}\,\mathrm{d}\nu \\
&\approx -\frac{16}{3}\frac{\sigma T^3}{\alpha_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}z} \\
&\approx -\frac{1}{3}\frac{c}{\kappa_\mathrm{R}\rho}\frac{\mathrm{d}u}{\mathrm{d}z}
\end{aligned}
$$

$$u = (4\sigma/c)T^4$$

$$l \equiv 1/\rho\kappa_\mathrm{R}$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

The Eddington approximation

$$
\begin{aligned}
I_\nu(\tau_\nu,\mu) &\approx S_\nu(\tau_\nu) + \mu \left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \\
J_\nu(\tau_\nu) &\approx S_\nu(\tau_\nu) \\
F_\nu(\tau_\nu) &\approx \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu} \\
K_\nu(\tau_\nu) &\approx \frac{1}{3} S_\nu(\tau_\nu).
\end{aligned}
$$

$$K_\nu(\tau_\nu) \approx \frac{1}{3} J_\nu(\tau_\nu)$$

$$
\begin{aligned}
K_\nu(\tau_\nu) &\equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu,\mu)\mu^2\,\mathrm{d}\mu \\
&\approx \frac{1}{2} < I_\nu(\tau_\nu,\mu) > \int_{-1}^{+1} \mu^2\,\mathrm{d}\mu \\
&\approx \frac{1}{3} J_\nu(\tau_\nu)
\end{aligned}
$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

The Eddington approximation

$$I_\nu(\tau_\nu, \mu) \equiv a_0(\tau_\nu) + a_1(\tau_\nu)\,\mu$$

$$J_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu)\,\mathrm{d}\mu = a_0(\tau_\nu),$$

$$H_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} \mu I_\nu(\tau_\nu, \mu)\,\mathrm{d}\mu = a_1(\tau_\nu)/3,$$

$$K_\nu(\tau_\nu) \equiv \frac{1}{2}\int_{-1}^{+1} \mu^2 I_\nu(\tau_\nu, \mu)\,\mathrm{d}\mu = a_0(\tau_\nu)/3$$

Plane-parallel Radiative Transfer

Approximate solutions

Approximations at large depth

Second order transport equation

$$\frac{1}{3}\frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

$$S_\nu = (1-\varepsilon_\nu)J_\nu + \varepsilon_\nu B_\nu$$

$$\frac{1}{3}\frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = \varepsilon_\nu \left[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)\right]$$

Atmospheres of Plane-parallel Stars

Classical modelling

Assumption

- The atmosphere is spherically symmetric.
- Element mixture homogenous with depth.
- Hydrostatic equilibrium.
- Statistical equilibrium / time independence.
- Atmosphere’s mass small relative to stellar.
- No sources or sinks of energy.
- Energy transport is radiative and convective.
- Maxwellian distribution for free particles.

Atmospheres of Plane-parallel Stars

Classical modelling

Model parameters

- Stellar Luminosity L
- Stellar Radius R
- Element mixture $[\mathrm{Fe/H}]$
- Microturbulence ξ_{micro}
- Effective temperature $T_{\mathrm{eff}} = (L/4\pi\sigma R^2)^{1/4}$
- Surface gravity $g_s = GM/R^2$

$$T_{\mathrm{eff}},\ \log g_s,\ [\mathrm{Fe/H}] \text{ and } \xi_{\mathrm{micro}}$$

Atmospheres of Plane-parallel Stars

Pressure stratification

Gas law

$$P_g V = n_{\rm mole} \mathcal{R} T$$

$$\mathcal{R} = k N_A = k / m_H$$

$$N_g = n_{\rm mole} N_A / V$$

$$\mu \equiv \overline{m} / m_H$$

$$\rho = N_g \mu \, m_H$$

$$\mathcal{R} = 8.314 \times 10^7 \,\mathrm{erg\,mole^{-1}\,K^{-1}}$$

$$k = 1.38 \times 10^{-16} \,\mathrm{erg\,K^{-1}}$$

$$N_A = 6.02 \times 10^{23} \,\mathrm{mole^{-1}}$$

$$m_H = 1.66 \times 10^{-24} \,\mathrm{g}$$

$$P_g = \frac{n_{\rm mole} N_A}{V} \frac{\mathcal{R}}{N_A} T = N_g k T = \frac{\rho k T}{\mu \, m_H} = \frac{\rho \mathcal{R} T}{\mu}$$

$$P_g = \sum_i P_i = \sum_i N_i k T$$

$$P_e = N_e k T$$

Atmospheres of Plane-parallel Stars

Pressure stratification

Particle densities

Chemical composition

E	$A_{\rm E}$	A_{12}	χ_0	χ_1	E	$A_{\rm E}$	A_{12}	χ_0	χ_1
H	1.000	12.0	13.60	–	Al	2.5×10^{-6}	6.4	5.99	18.83
He	7.9×10^{-2}	10.9	24.59	54.42	Si	3.2×10^{-5}	7.5	8.15	16.35
C	3.2×10^{-4}	8.5	11.26	24.38	S	1.6×10^{-5}	7.2	10.36	23.33
N	1.0×10^{-4}	8.0	14.53	29.60	K	1.0×10^{-7}	5.0	4.34	31.63
O	6.3×10^{-4}	8.8	13.62	35.12	Ca	2.0×10^{-6}	6.3	6.11	11.87
Na	2.0×10^{-6}	6.3	5.14	47.29	Cr	7.9×10^{-7}	5.9	6.77	16.50
Mg	2.5×10^{-5}	7.4	7.65	15.04	Fe	4.0×10^{-5}	7.6	7.87	16.16

$$A_{12} \equiv \log N_{\rm E} - \log N_{\rm H} + 12$$

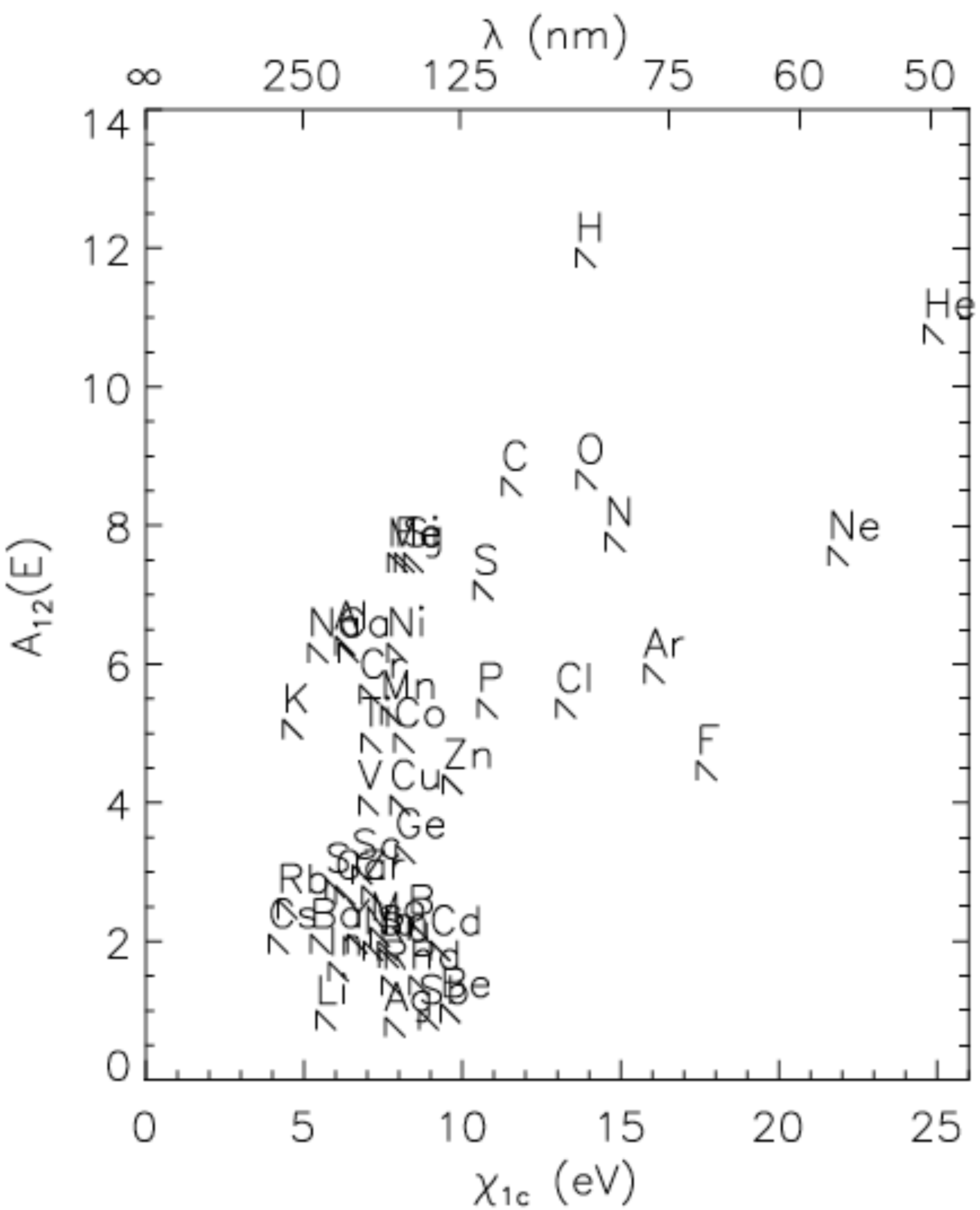
$$X = 0.73 \quad Y = 0.25 \quad Z = 0.017$$

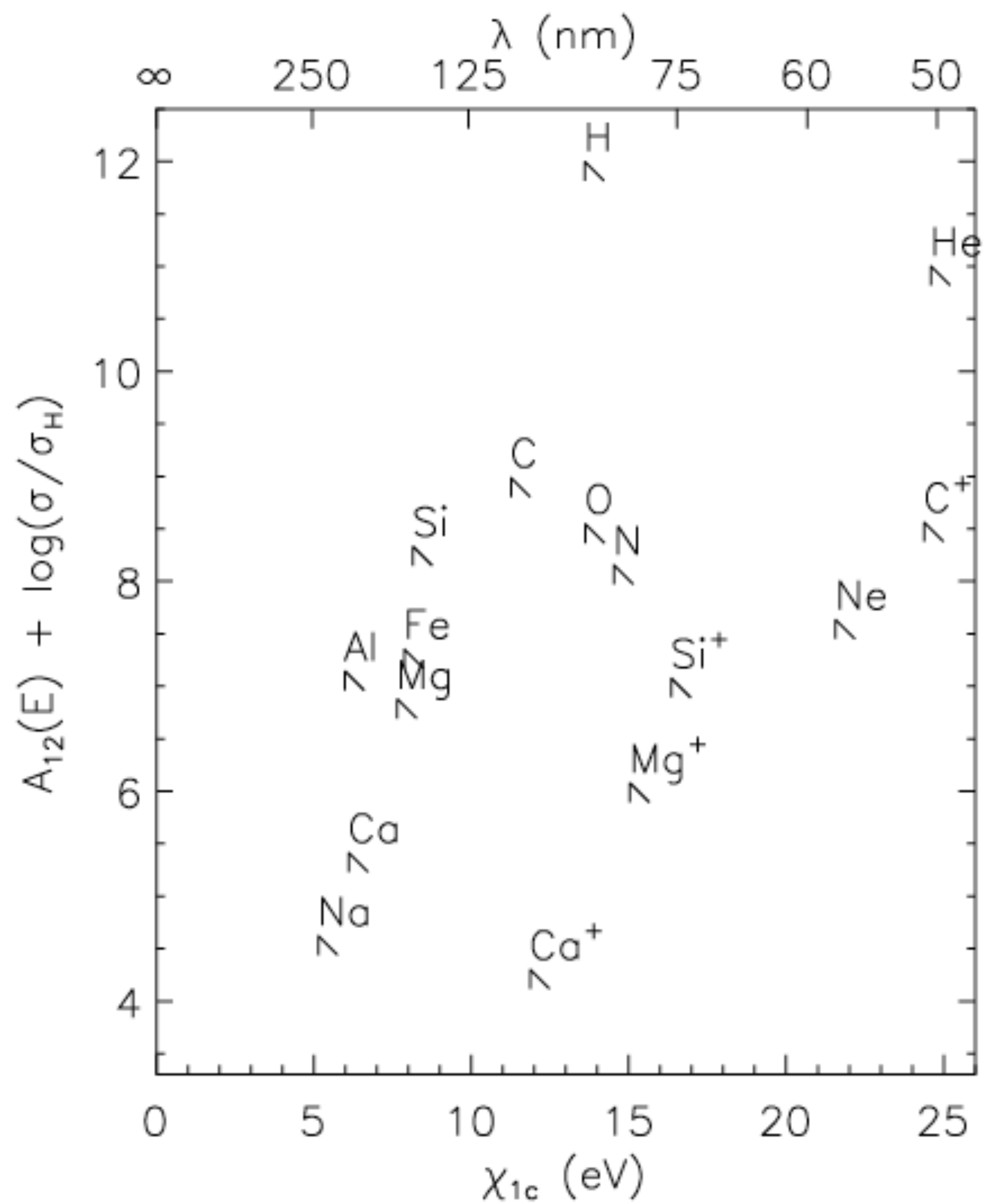
Atmospheres of Plane-parallel Stars

Pressure stratification

Particle densities

Electron donors





Atmospheres of Plane-parallel Stars

Pressure stratification

Particle densities

Electron donors

$$A_{\rm M} = N_{\rm M}/N_{\rm H} \ll 1$$

$$N_{\rm g} = N_{\rm H} + A_{\rm M} N_{\rm H} + f_{\rm H} N_{\rm H} + f_{\rm M} A_{\rm M} N_{\rm H}$$

$$N_{\rm e} = f_{\rm H} N_{\rm H} + f_{\rm M} A_{\rm M} N_{\rm H}$$

$$\frac{N_{\rm e}}{N_{\rm g}} = \frac{f_{\rm H} + f_{\rm M} A_{\rm M}}{1 + f_{\rm H} + (1 + f_{\rm M}) A_{\rm M}}$$

$$f_{\rm H} \approx 1 \quad \rightarrow \quad \frac{N_{\rm e}}{N_{\rm g}} \approx \frac{1}{2}$$

$$A_{\rm M} \ll f_{\rm H} \ll 1 \quad \rightarrow \quad \frac{N_{\rm e}}{N_{\rm g}} \approx f_{\rm H}$$

$$f_{\rm H} \approx 0 \quad \rightarrow \quad \frac{N_{\rm e}}{N_{\rm g}} \approx f_{\rm M} A_{\rm M}$$

Atmospheres of Plane-parallel Stars

Pressure stratification

Particle densities

Electron and gas pressure

$$P_{\rm g} = N_{\rm g} kT$$

$$P_{\rm e} = N_{\rm e} kT$$

$$N_{\rm z} = N_{\rm I} + N_{\rm II} + N_{\rm III}$$

$$\frac{1}{f_{\rm II}} = \frac{N_{\rm I}}{N_{\rm II}} + \frac{N_{\rm II}}{N_{\rm II}} + \frac{N_{\rm III}}{N_{\rm II}}$$

$$\frac{1}{f_{\rm III}} = \frac{N_{\rm I}}{N_{\rm II}} \frac{N_{\rm II}}{N_{\rm III}} + \frac{N_{\rm II}}{N_{\rm III}} + \frac{N_{\rm III}}{N_{\rm III}}$$

$$E = \frac{N_{\rm e}}{N_{\rm nuclei}} = \frac{\sum_{\rm z} N_{\rm z} f_{\rm II}(z) + 2 \sum_{\rm z} N_{\rm z} f_{\rm III}(z)}{\sum_{\rm z} N_{\rm z}}$$

$$\frac{P_{\rm g}}{P_{\rm e}} = \frac{(N_{\rm ions} + N_{\rm atoms} + N_{\rm e})\ kT}{N_{\rm e} kT} = \frac{(N_{\rm nuclei} + N_{\rm e})\, kT}{N_{\rm e} kT} = \frac{E+1}{E}$$

Atmospheres of Plane-parallel Stars

Pressure stratification

Hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -g\rho$$

$$\frac{\mathrm{d}P}{\mathrm{d}\tau_0} = \frac{g}{\kappa_0}$$

$$\mathrm{d}\tau_0 = -\kappa_0 \rho \, \mathrm{d}z$$

$$\frac{\mathrm{d}p}{\mathrm{d}\tau_0} = \frac{4\pi}{c} \int_0^\infty \frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_0} \, \mathrm{d}\nu = \frac{4\pi}{c} \int_0^\infty \frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} \frac{\mathrm{d}\tau_\nu}{\mathrm{d}\tau_0} \, \mathrm{d}\nu = \frac{1}{c} \int_0^\infty \mathcal{F}_\nu \frac{\kappa_\nu}{\kappa_0} \, \mathrm{d}\nu$$

Model completion

$$P_{\mathrm{g}}^{1/2} \frac{\mathrm{d}P_{\mathrm{g}}}{\mathrm{d}\tau_0} = P_{\mathrm{g}}^{1/2} \frac{g}{\kappa_0}$$

$$P_{\mathrm{g}}(\tau_0) = \left(\frac{3g}{2} \int_0^{\tau_0} \frac{P_{\mathrm{g}}^{1/2}(t_0)}{\kappa_0(t_0)} \, \mathrm{d}t_0 \right)^{2/3}$$

Atmospheres of Plane-parallel Stars

Pressure stratification

Plane-parallel layers

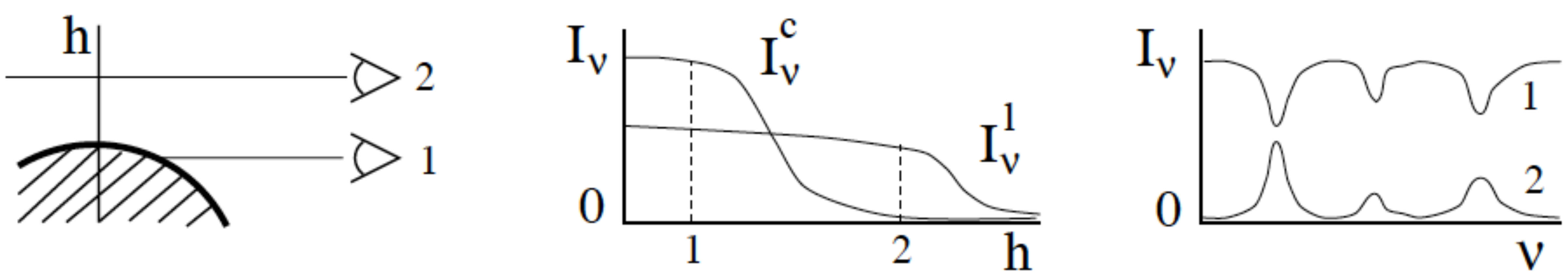
$$H_\mathrm{P} \equiv \mathcal{R}T/\mu g$$

$$\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}z} = -\frac{\mu g}{\mathcal{R}T}P_\mathrm{g} = -\frac{P_\mathrm{g}}{H_\mathrm{P}}$$

$$P_\mathrm{g}(z) = P_\mathrm{g}(0)\,\mathrm{e}^{-z/H_\mathrm{P}}$$

$$\frac{H_\mathrm{P}}{R_*} = \frac{\mathcal{R}T}{\mu g R_*} = \frac{\mathcal{R}T R_*}{\mu G M_*} = 4.4\times10^{-8}\,\frac{T_\mathrm{eff}\,(R_*/R_\odot)}{\mu\,(M_*/M_\odot)} \ll 1$$

Solar limb

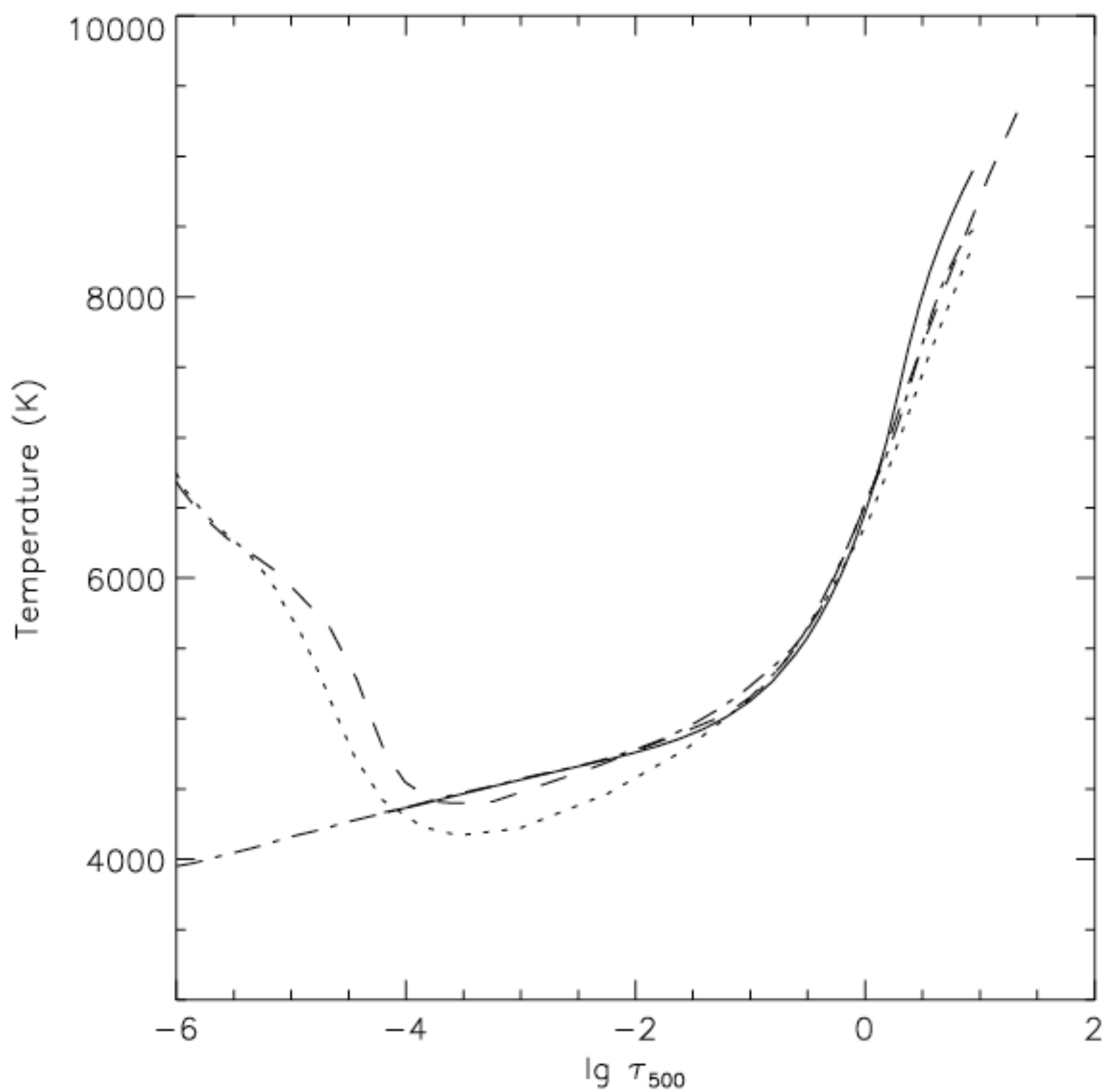


Atmospheres of Plane-parallel Stars

Temperature stratification

Empirical models

I_ν ➧ $S_\nu(\tau_\nu = \mu)$ ➧ $T(\tau_0)$



Atmospheres of Plane-parallel Stars

Temperature stratification

Center-limb variation

$$\frac{I_\nu(0,\mu)}{I_\nu(0,1)} = a_\nu + b_\nu \mu + c_\nu \left(1 - \mu \, \ln(1 + \frac{1}{\mu})\right)$$

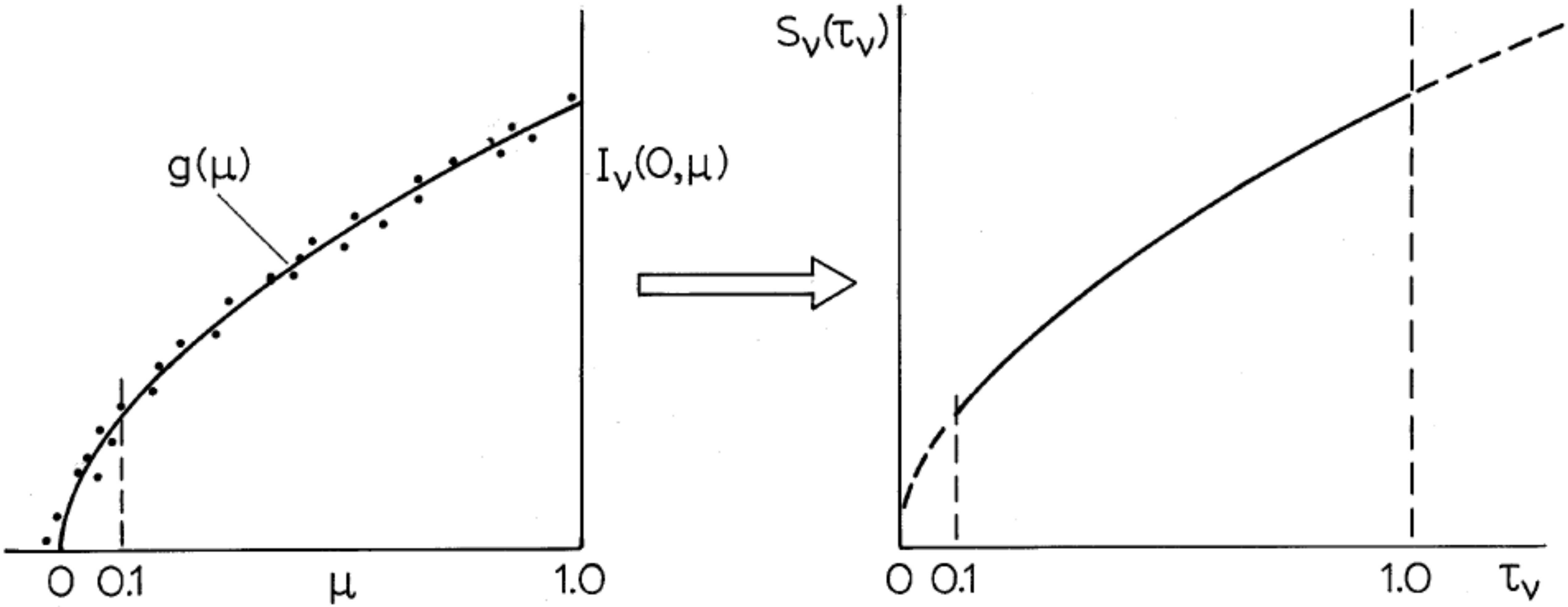
$$S_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu + c_\nu E_2(\tau_\nu)$$

$$\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\tau_0} = \frac{\kappa_\nu \rho \, \mathrm{d}z}{\kappa_0 \rho \, \mathrm{d}z} = \frac{\kappa_\nu}{\kappa_0} \qquad \tau_\nu(\tau_0) = \int_0^{\tau_0} \frac{\kappa_\nu}{\kappa_0} \, \mathrm{d}t_\nu$$

Atmospheres of Plane-parallel Stars

Temperature stratification

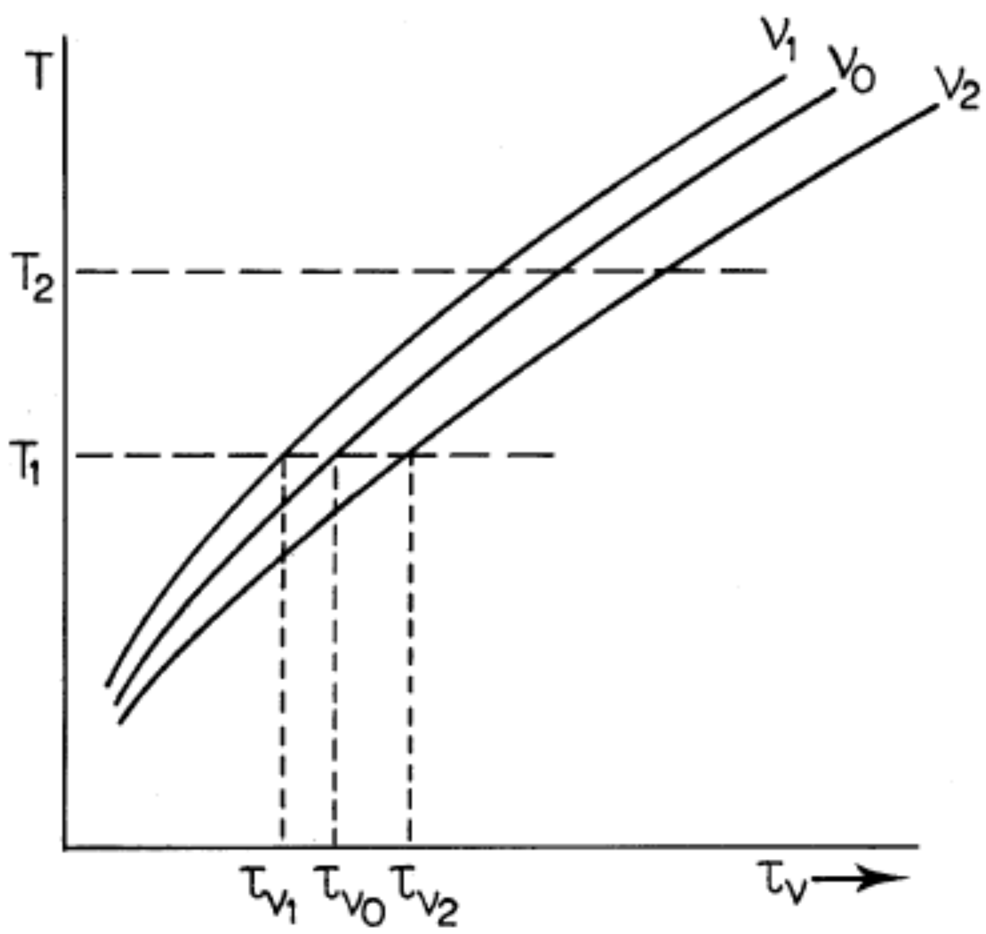
Center-limb variation

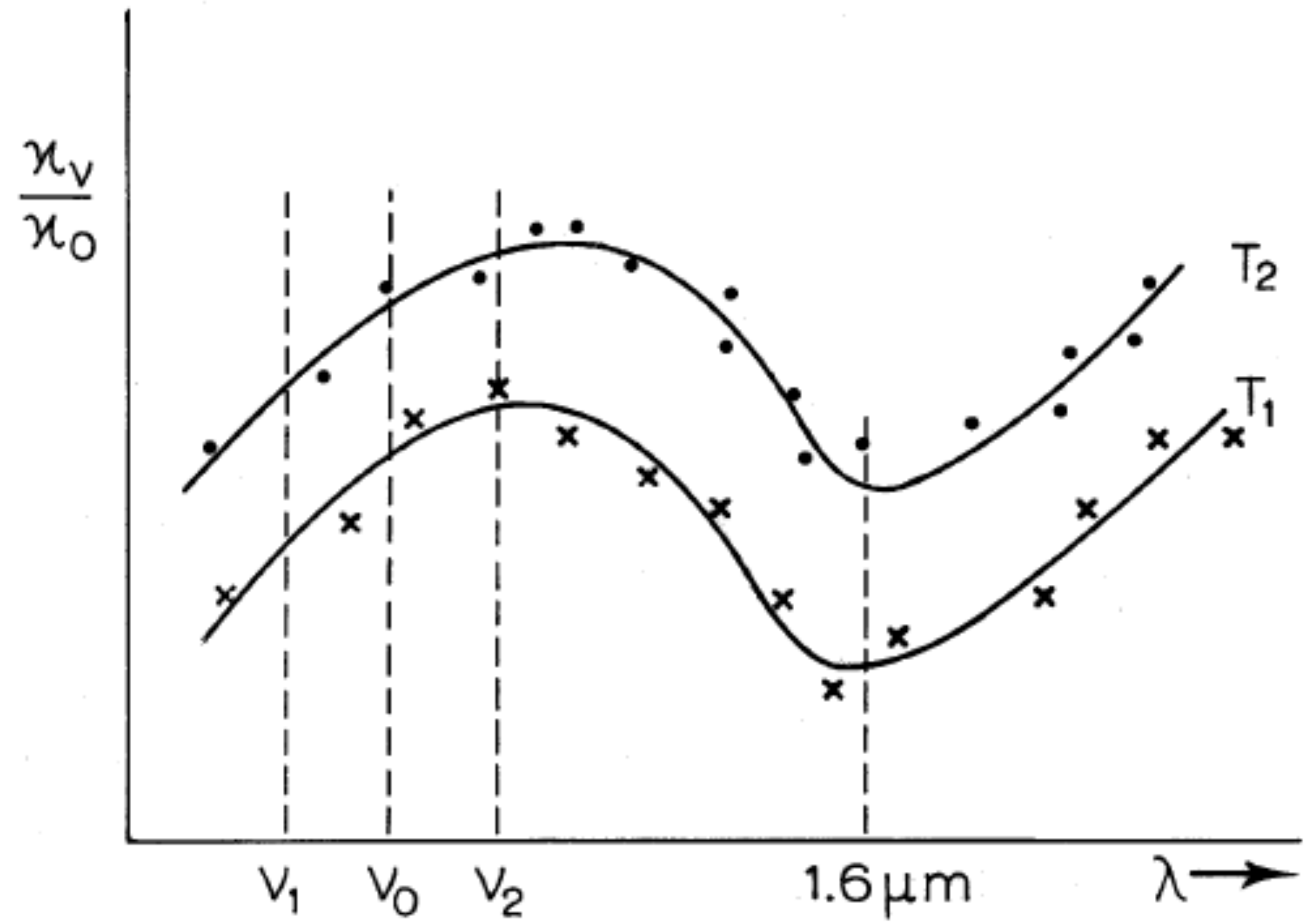


Atmospheres of Plane-parallel Stars

Temperature stratification

Center-limb variation





Atmospheres of Plane-parallel Stars

Temperature stratification

Flux constancy

$$\nabla \cdot \mathbf{F}_{\rm tot}(\mathbf{r}) = \nabla \cdot [\mathbf{F}_{\rm rad}(\mathbf{r}) + \mathbf{F}_{\rm conv}(\mathbf{r}) + \mathbf{F}_{\rm cond}(\mathbf{r}) + \mathbf{F}_{\rm mech}(\mathbf{r})] \equiv 0.$$

$$\frac{\mathrm{d}F_{\rm tot}}{\mathrm{d}z} = 0$$

Radiative equilibrium

$$\mathcal{F}_{\rm rad}(z) \equiv \int_0^\infty \mathcal{F}_\nu(z)\,\mathrm{d}\nu = \mathcal{F}$$

$$\mathcal{F} \equiv \sigma T_{\rm eff}^4 = \frac{L_*}{4\pi R_*^2}$$

$$\frac{\mathrm{d}\mathcal{F}_{\rm rad}(z)}{\mathrm{d}z} = 0$$

Atmospheres of Plane-parallel Stars

Temperature stratification

Radiative equilibrium

Strömgren equation

$$\int_0^\infty \kappa_\nu(z)\rho(z)\, J_\nu(z)\, \mathrm{d}\nu = \int_0^\infty \kappa_\nu(z)\rho(z)\, S_\nu(z)\, \mathrm{d}\nu$$

Total radiative flux divergence

$$\Phi_{\mathrm{tot}}(z) \equiv \frac{\mathrm{d}\mathcal{F}_{\mathrm{rad}}(z)}{\mathrm{d}z} = 4\pi \int_0^\infty \alpha_\nu(z)\,[S_\nu(z) - J_\nu(z)]\; \mathrm{d}\nu = 0 \quad \mathrm{erg\,cm^{-3}\,s^{-1}}$$

$$\Phi_{\mathrm{tot}}(z) \equiv \frac{\mathrm{d}\mathcal{F}_{\mathrm{rad}}(z)}{\mathrm{d}z} = \frac{1}{2}\int_0^\infty \int_{-1}^{+1} [j_{\nu\mu}(z) - \alpha_{\nu\mu}(z)\, I_{\nu\mu}(z)]\; \mathrm{d}\mu\, \mathrm{d}\nu = 0$$

Atmospheres of Plane-parallel Stars

Temperature stratification

Line cooling

$$\begin{aligned}
\Phi_{ul} &= 4\pi\alpha^l_{\nu_0}(S^l_{\nu_0} - \overline{J}_{\nu_0}) \\
&= 4\pi j^l_{\nu_0} - 4\pi\alpha^l_{\nu_0}\overline{J}_{\nu_0} \\
&= h\nu_0 \left[n_u(A_{ul} + B_{ul}\overline{J}_{\nu_0}) - n_l B_{lu}\overline{J}_{\nu_0}\right] \\
&= h\nu_0 \left[n_u R_{ul} - n_l R_{lu}\right],
\end{aligned}$$

$$\int \varphi(\nu-\nu_0)\,\mathrm{d}\nu = \int \chi(\nu-\nu_0)\,\mathrm{d}\nu = \int \psi(\nu-\nu_0)\,\mathrm{d}\nu = 1$$

Wien limit

$$\Phi_{ul} = h\nu_0 \left[n_u R_{ul} - n_l R_{lu}\right] \approx 4\pi\, b_u \left[\alpha^l_{\nu_0}\right]_{\mathrm{LTE}} \left(B_{\nu_0} - \frac{b_l}{b_u}\overline{J}_{\nu_0}\right)$$

Atmospheres of Plane-parallel Stars

Temperature stratification

Continuum cooling

$$\Phi_{ci} = 4\pi\, n_i^{\rm LTE} b_c \int_{\nu_0}^{\infty} \sigma_{ic}(\nu) \left[B_\nu \left(1 - \,{\rm e}^{-h\nu/kT}\right) - \frac{b_i}{b_c} J_\nu \left(1 - \frac{b_c}{b_i}\,{\rm e}^{-h\nu/kT}\right)\right] {\rm d}\nu$$

Wien limit

$$\Phi_{ci} = 4\pi\, n_i^{\rm LTE} b_c \int_{\nu_0}^{\infty} \sigma_{ic}(\nu) \left(B_\nu - \frac{b_i}{b_c} J_\nu \right) {\rm d}\nu.$$

Atmospheres of Plane-parallel Stars

The grey approximation

$$\int_0^\infty \mu \frac{\mathrm{d}I_\nu(\tau_\nu,\mu)}{\mathrm{d}\tau_\nu}\,\mathrm{d}\nu = \int_0^\infty \left[I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)\right]\,\mathrm{d}\nu$$

$$\mu \frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = I(\tau,\mu) - S(\tau).$$

$$S(\tau) = J(\tau)$$

$$J(\tau) = \mathbf{\Lambda}_\tau[S(t)]$$

$$F(\tau) = \mathbf{\Phi}_\tau[S(t)] = F$$

Atmospheres of Plane-parallel Stars

The grey approximation

Grey RE source function

$$S(\tau) \approx c\,(1 + \frac{3}{2}\tau)$$

$$F = \boldsymbol{\Phi}_\tau[S(t)] = \frac{\mathrm{d}}{\mathrm{d}\tau}\chi_\tau[S(\tau)] = 4\frac{\mathrm{d}K(\tau)}{\mathrm{d}\tau}$$

$$K(\tau) = (1/4)F\tau + a$$

$$K(\tau) \approx (1/3)J(\tau) = (1/3)S(\tau)$$

$$S(\tau) \approx (3/4)F\tau + 3a$$

$$S(\tau) = \frac{3}{4}(\tau + q(\tau))\,F$$

$$\tau + q(\tau) = \boldsymbol{\Lambda}_\tau[\tau + q(\tau)]$$

$$S(0) \;=\; J(0) \;\approx\; F/2$$

$$q(\tau) = 2/3$$

$$S(\tau) \approx \frac{3}{4}(\tau + \frac{2}{3})\,F = (\frac{3}{4}\tau + \frac{1}{2})\,F = \frac{1}{2}(1 + \frac{3}{2}\tau)\,F$$

$$F \;=\; (\sigma/\pi)\,T_{\rm eff}^4$$

Atmospheres of Plane-parallel Stars

The grey approximation

Grey RE temperature stratification

$$S(\tau) = B(\tau) = (\sigma/\pi)\,T^4$$

$$T(\tau) \approx T_{\rm eff}\left(\frac{3}{4}\tau + \frac{1}{2}\right)^{1/4}$$

$$T_{\rm eff} = T(\tau = 2/3)$$

$$S_\nu(\tau) = B_\nu\,[T(\tau)]$$

Atmospheres of Plane-parallel Stars

The grey approximation

Grey RE scattering

$$S_\nu = (1 - \varepsilon_\nu) J_\nu + \varepsilon_\nu B_\nu$$

$$\begin{aligned}
\int_0^\infty \kappa_\nu \, \rho \, J_\nu \, \mathrm{d}\nu &= \int_0^\infty \kappa_\nu \, \rho \, S_\nu \, \mathrm{d}\nu \\
\kappa \, \rho \, J &= \kappa \, \rho \, [(1 - \varepsilon) J + \varepsilon B] \\
\varepsilon J &= \varepsilon B \\
J &= B = S,
\end{aligned}$$

Atmospheres of Plane-parallel Stars

The grey approximation

Grey RE limb darkening

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}(\mu + \frac{2}{3})$$

		Limb darkening $I(0,\mu)/I(0,1)$				
$r/R_\odot$	μ	Observed	Radiative equilibrium		Convective equilibrium	
0.00	1.00	1.00	1.00	1.00	1.00	1.00
0.20	0.98	0.99	0.99	0.99	0.98	0.97
0.40	0.92	0.97	0.95	0.95	0.92	0.87
0.60	0.80	0.92	0.87	0.88	0.80	0.70
0.80	0.60	0.81	0.73	0.76	0.60	0.44
0.90	0.44	0.70	0.63	0.66	0.44	0.27
0.98	0.20	0.49	0.47	0.52	0.20	0.08
1.00	0.00	≈ 0.40	0.33	0.40	0.00	0.00

Atmospheres of Plane-parallel Stars

The grey approximation

Grey extinction and mean extinction

$$4\frac{\mathrm{d}K_\nu(z)}{\mathrm{d}\tau_\nu} = 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z} = F_\nu(z)$$

$$\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu \equiv \frac{1}{\overline{\kappa}(z)}\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu$$

$$\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\kappa_\nu(z)\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu = \int_0^\infty F_\nu(z)\,\mathrm{d}\nu = F(z)$$

$$\overline{\kappa}(z) \equiv \frac{\int_0^\infty \kappa_\nu(z)\,F_\nu(z)\,\mathrm{d}\nu}{\int_0^\infty F_\nu(z)\,\mathrm{d}\nu} = \int_0^\infty \kappa_\nu(z)\,\frac{F_\nu(z)}{F(z)}\,\mathrm{d}\nu$$

$$\frac{1}{\overline{\kappa}(z)}\int_0^\infty 4\frac{\mathrm{d}K_\nu(z)}{-\rho(z)\,\mathrm{d}z}\,\mathrm{d}\nu = \frac{1}{\overline{\kappa}(z)}\,4\frac{\mathrm{d}K(z)}{-\rho(z)\,\mathrm{d}z}$$

$$\overline{\frac{1}{\kappa(z)}} \equiv \frac{\int_0^\infty [1/\kappa_\nu(z)]\,(\mathrm{d}K_\nu(z)/\mathrm{d}z)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}K_\nu(z)/\mathrm{d}z)\,\mathrm{d}\nu} = \int_0^\infty \frac{1}{\kappa_\nu(z)}\,\frac{\mathrm{d}K_\nu(z)/\mathrm{d}z}{\mathrm{d}K(z)/\mathrm{d}z}\,\mathrm{d}\nu$$

Atmospheres of Plane-parallel Stars

The grey approximation

Flux-weighted mean and Rosseland mean

$$K_\nu \approx \frac{1}{3} J_\nu \approx \frac{1}{3} B_\nu$$

$$\mathrm{d}K_\nu/\mathrm{d}z \approx (1/3)\,\mathrm{d}B_\nu/\mathrm{d}z = (1/3)\,(\mathrm{d}B_\nu/\mathrm{d}T)\,(\mathrm{d}T/\mathrm{d}z)$$

$$\frac{1}{\overline{\kappa}} \approx \int_0^\infty \frac{1}{\kappa_\nu} \frac{\mathrm{d}B_\nu/\mathrm{d}T}{\mathrm{d}B/\mathrm{d}T}\,\mathrm{d}\nu \equiv \frac{1}{\kappa_\mathrm{R}}$$

$$T(\tau_\mathrm{R}) = T_\mathrm{eff}\left[\frac{3}{4}\tau_\mathrm{R} + \frac{3}{4}q(\tau_\mathrm{R})\right]^{1/4}$$

$$\mathrm{d}\tau_\mathrm{R} = -\kappa_\mathrm{R}\rho\,\mathrm{d}z$$

$$J(\tau_\mathrm{R}) = S(\tau_\mathrm{R}) = B(\tau_\mathrm{R}) = \frac{\sigma}{\pi}T^4(\tau_\mathrm{R}) = \frac{3}{4}\left[\tau_\mathrm{R} + q(\tau_\mathrm{R})\right]\,F$$

$$q(\tau_\mathrm{R}) \approx 2/3$$

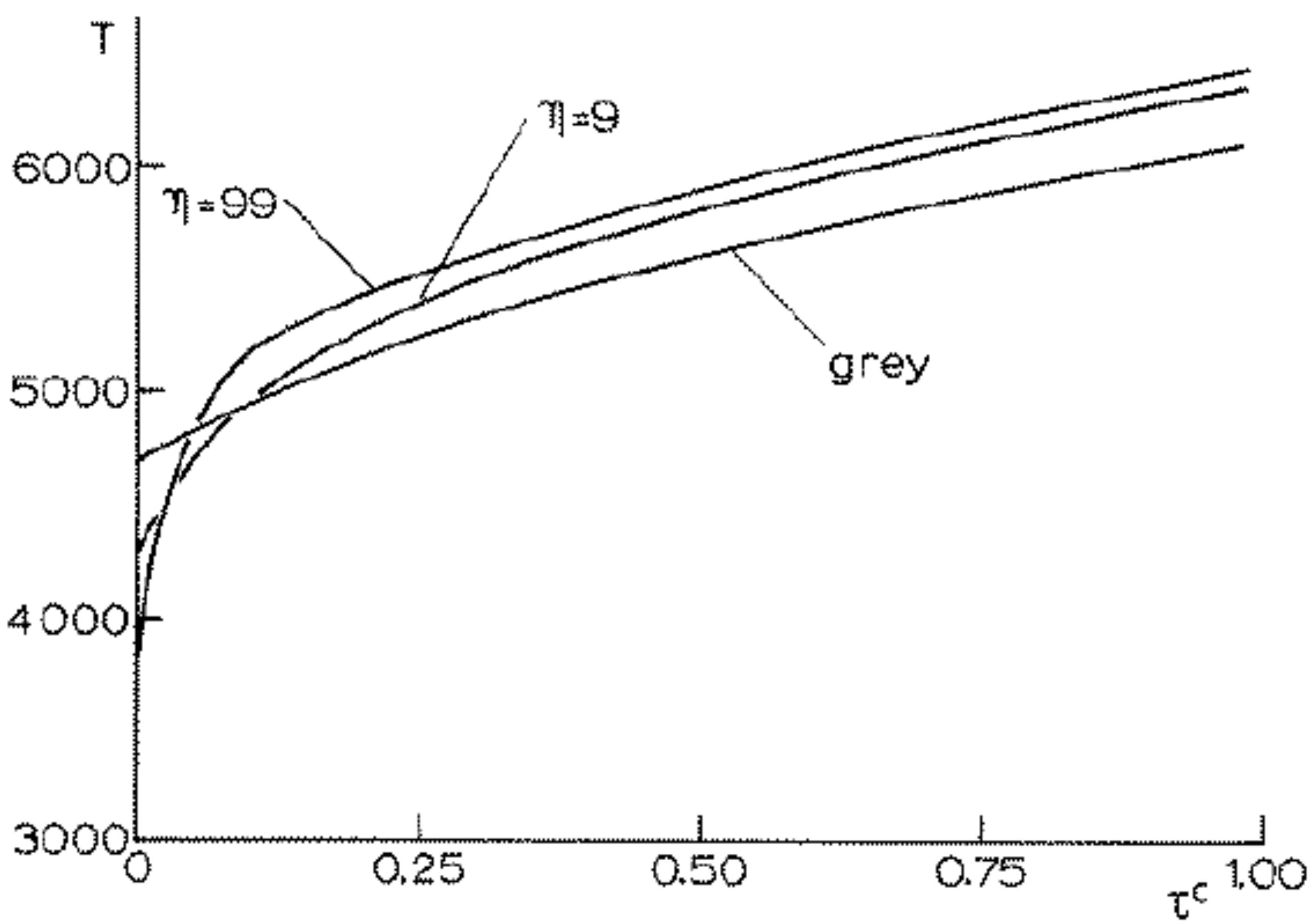
Atmospheres of Plane-parallel Stars

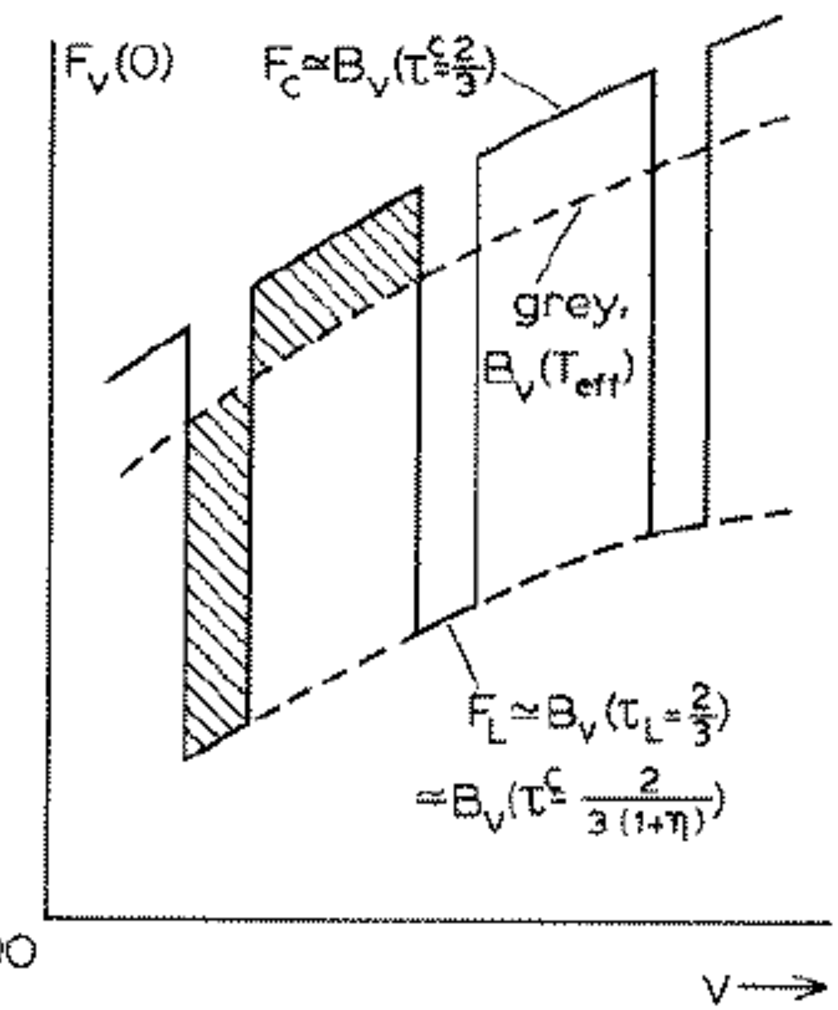
Line blanketing

Backwarming

$$\frac{\int_0^\infty F'_\nu \, d\nu}{\int_0^\infty F_\nu \, d\nu} = \frac{F'}{F} = \frac{(\sigma/\pi)\, {T'_{\rm eff}}^4}{(\sigma/\pi)\, {T_{\rm eff}}^4} = 1 - f$$

$$T_{\rm eff} = (1-f)^{-1/4}\, T'_{\rm eff} \approx (1 + f/4)\, T'_{\rm eff}$$





Atmospheres of Plane-parallel Stars

Line blanketing

Surface effects

$$\Phi_{ul}(z) = 4\pi\, \alpha^{l}_{\nu_0}(z) \left[S^{l}_{\nu_0}(z) - \overline{J}_{\nu_0}(z) \right]$$

$J_\nu(0) > B_\nu(0)$ high-frequency side

$J_\nu(0) < B_\nu(0)$ low-frequency side

Strong LTE lines

$$\mathrm{d}\tau^{\mathrm{tot}}_\nu = \mathrm{d}\tau^c_\nu + \mathrm{d}\tau^l_\nu = (1+\eta_\nu)\,\mathrm{d}\tau^c_\nu \qquad \eta = \kappa^l_\nu/\kappa^c_\nu$$

$$\tau^c_\nu = 1/(1+\eta_\nu)$$

$$J_\nu(0) = (1/2)B_\nu$$

$$\int_0^\infty \kappa_\nu(0)\, J_\nu(0)\, \mathrm{d}\nu = \int_0^\infty \kappa_\nu(0)\, B_\nu(0)\, \mathrm{d}\nu$$

Atmospheres of Plane-parallel Stars

Line blanketing

Strong scattering lines

$$J_{\nu_0}(0) \approx S_{\nu_0}(0) \approx \sqrt{\varepsilon_{\nu_0}} B_{\nu_0}(0) \ll B_{\nu_0}(0)$$

$$S^l_{\nu_0}(0) - \overline{J}_{\nu_0}(0) \approx \frac{\varepsilon_{\nu_0}}{1 + \sqrt{\varepsilon_{\nu_0}}} B_{\nu_0} \approx \varepsilon_{\nu_0} B_{\nu_0}$$

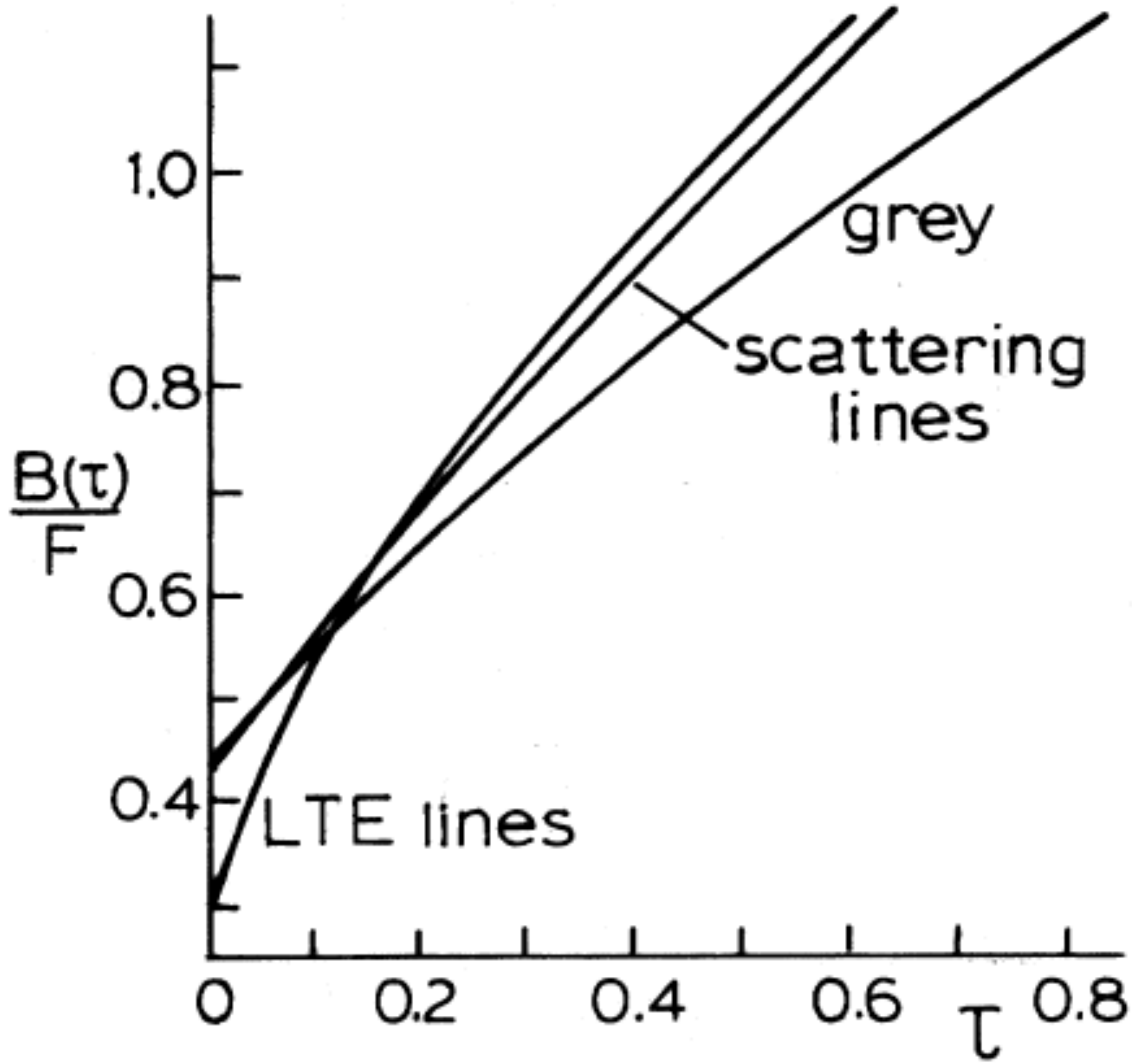
$$\begin{aligned}
\Phi_{ul} &= 4\pi \alpha^l_{\nu_0} (S^l_{\nu_0} - \overline{J}_{\nu_0}) \\
&= 4\pi\, \alpha^l_{\nu_0} \left[(1 - \varepsilon_{\nu_0}) \overline{J}_{\nu_0} + \varepsilon_{\nu_0} B_{\nu_0} - \overline{J}_{\nu_0} \right] \\
&= 4\pi\, \alpha^l_{\nu_0}\, \varepsilon_{\nu_0} \left(B_{\nu_0} - \overline{J}_{\nu_0} \right)
\end{aligned}$$

Scattering continua

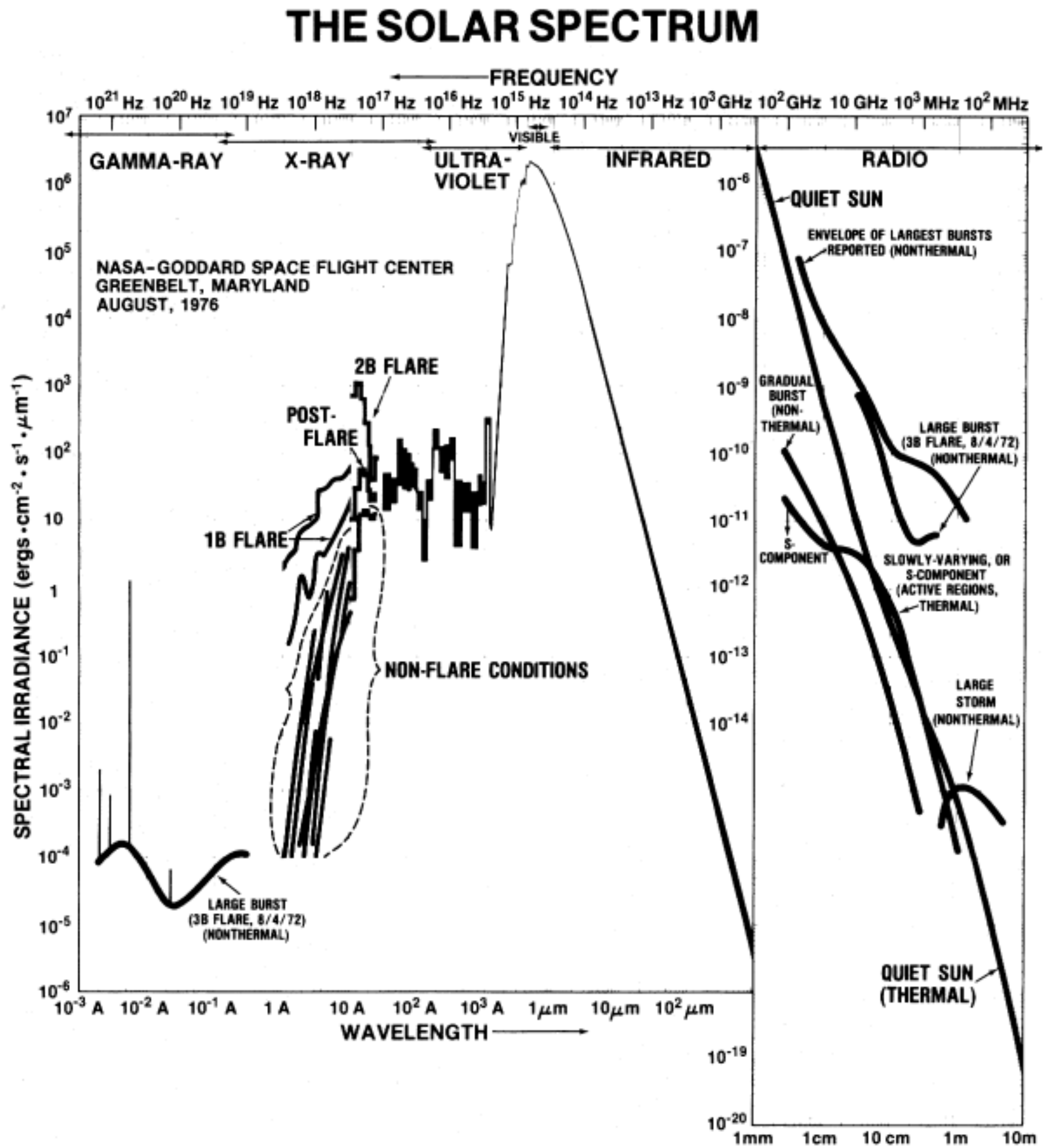
$$S_\nu = B_\nu$$

Atmospheres of Plane-parallel Stars

Line blanketing



Continua from Plane-parallel Stars



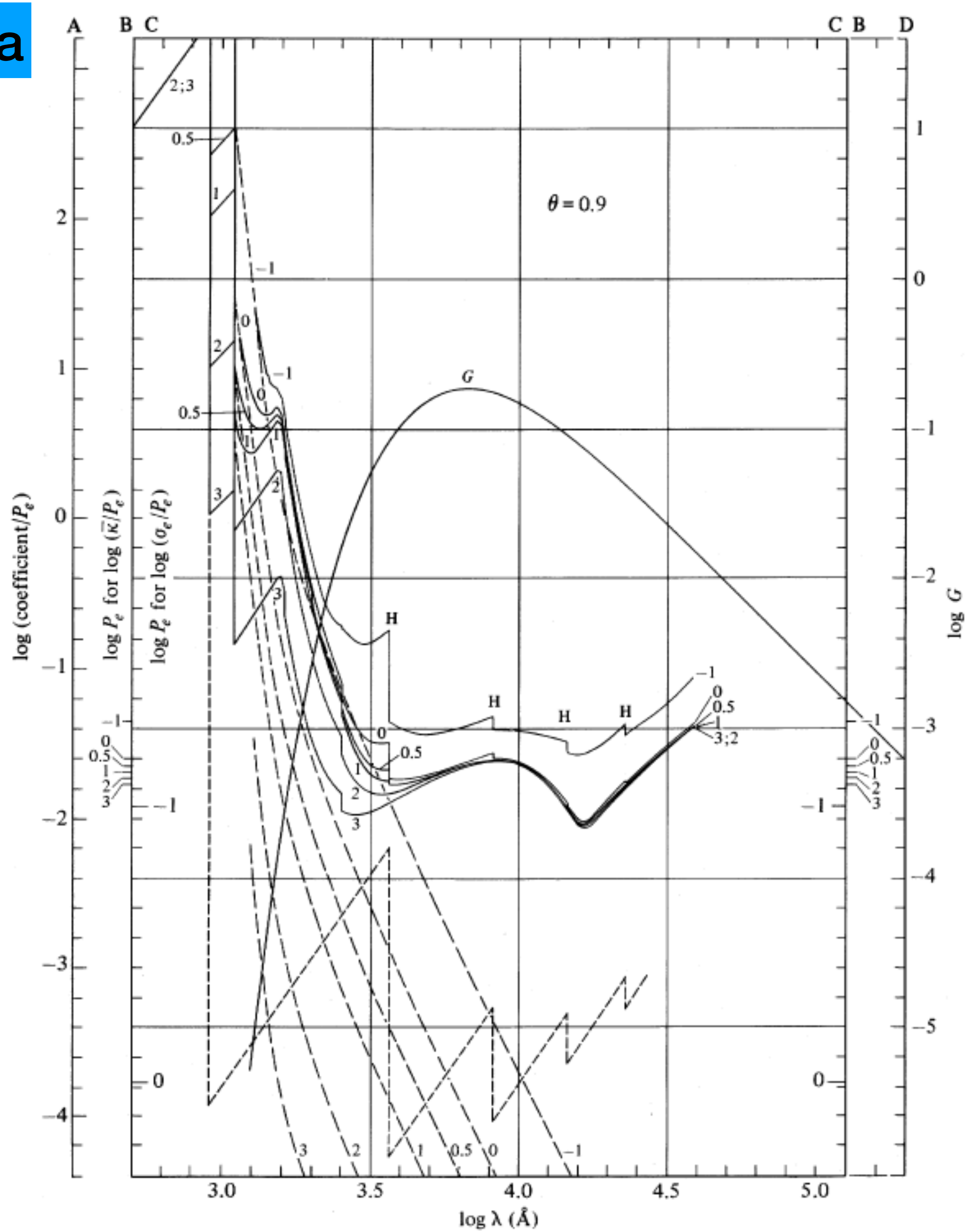
Continua from Plane-parallel Stars

Solar continua

Continuous extinction

- Free-free transitions.
- Bound-free transitions.
- Cyclotron radiation, synchrotron radiation, plasma radiation.
- Thomson scattering.
- Rayleigh scattering.
- Line haze.

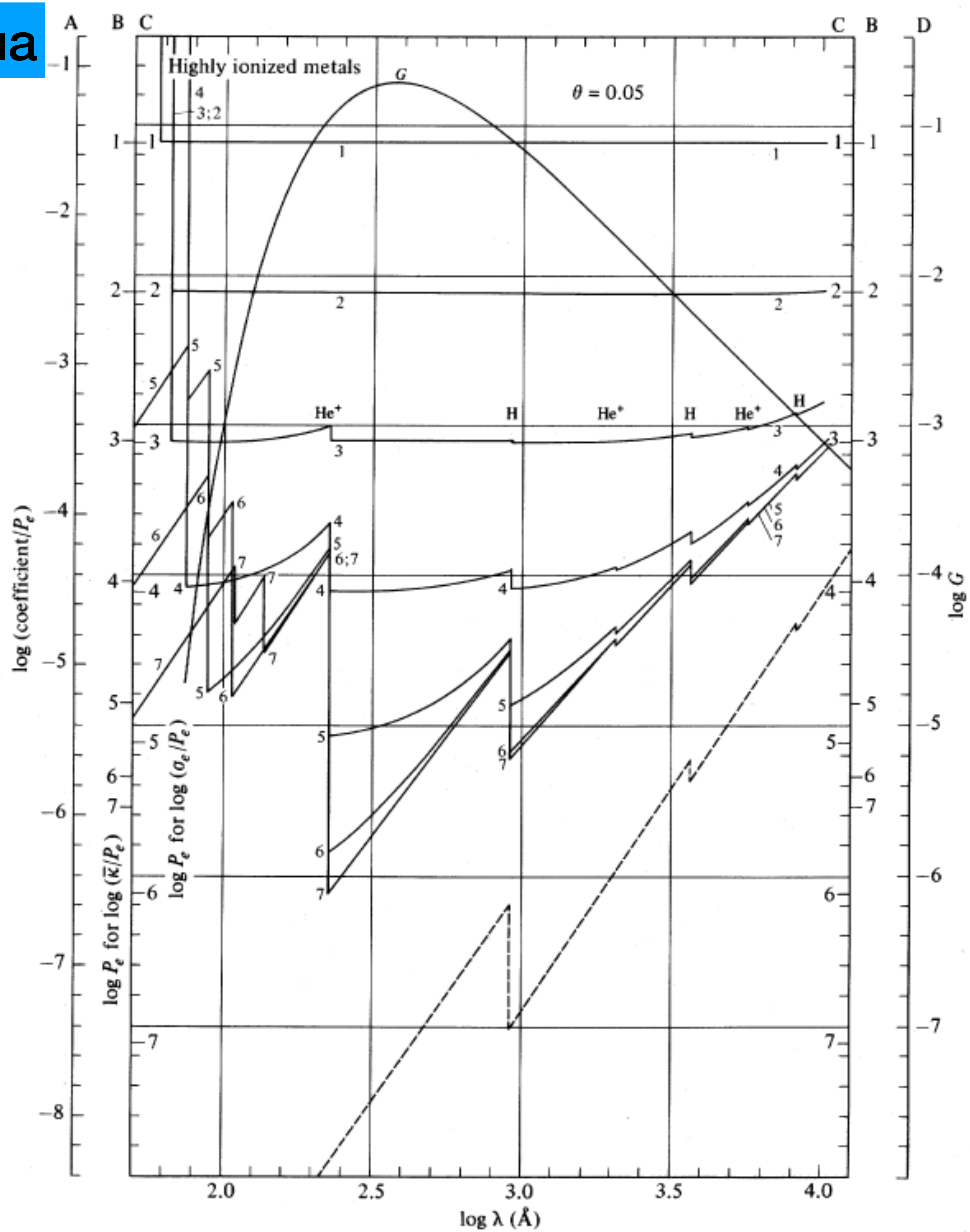
Solar continua



$$G \equiv \frac{dB_\nu/dT}{dB/dT}$$

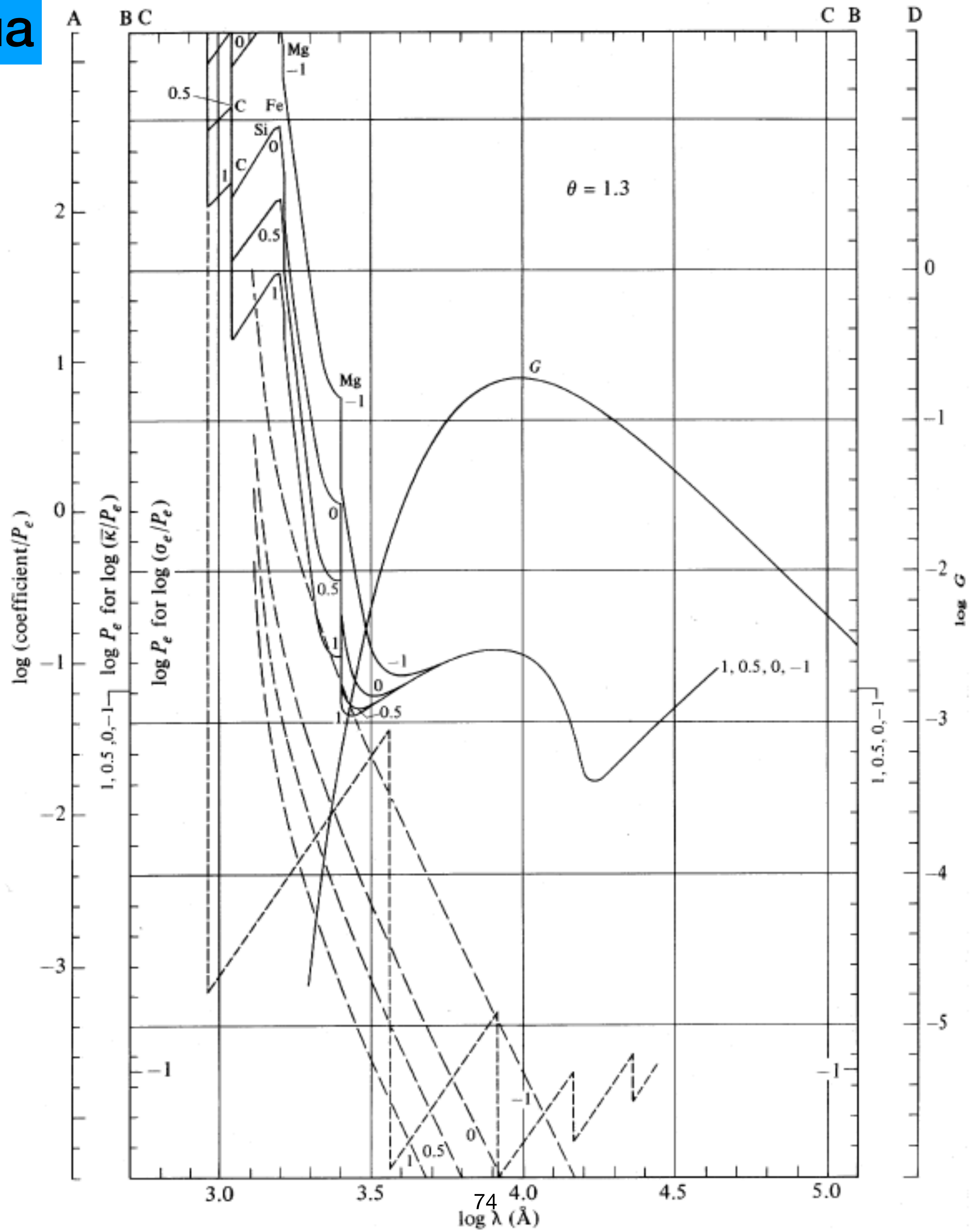
Stellar continua

O Star

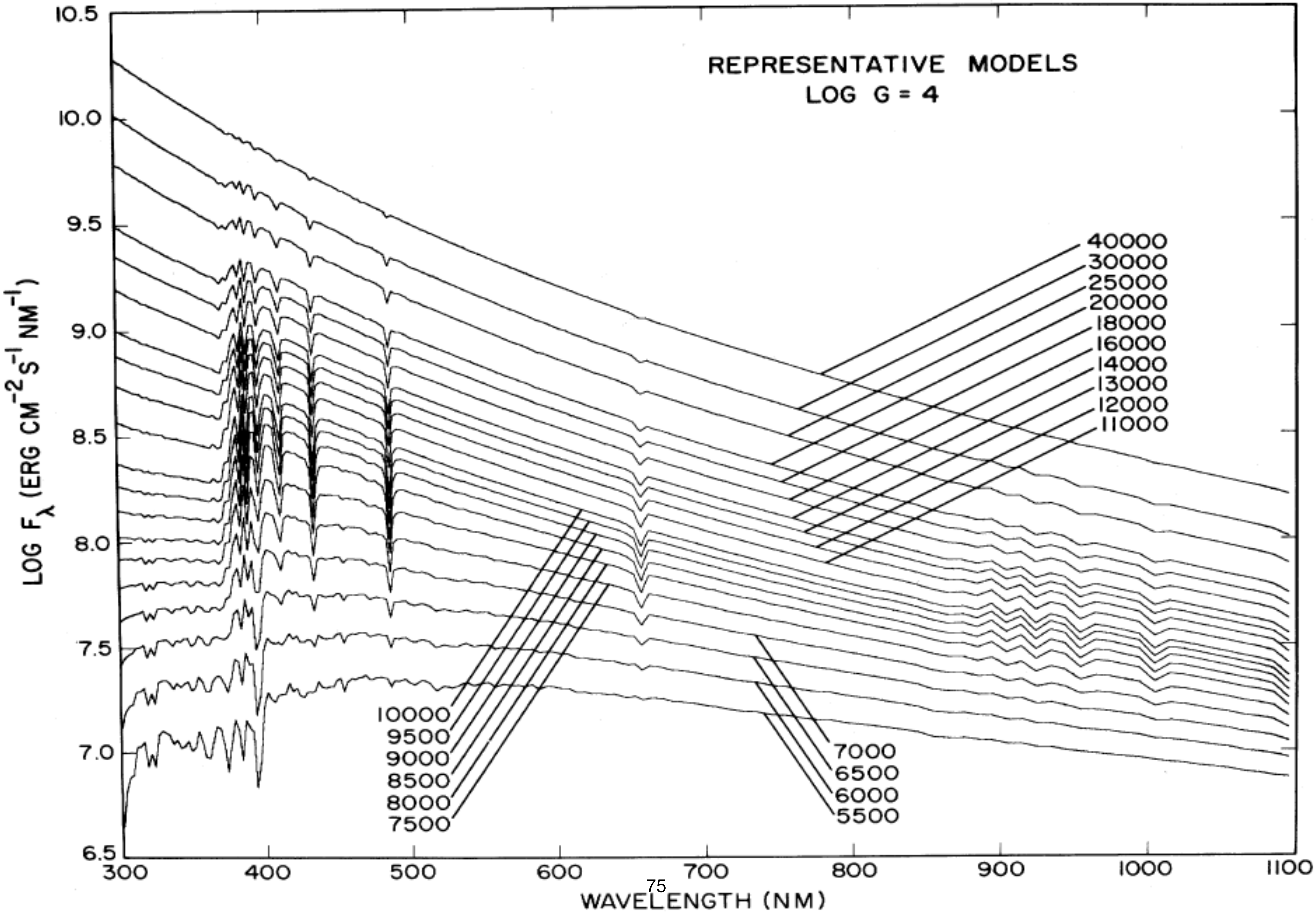


Stellar continua

K8 Star



Continua from Plane-parallel Stars



Continua from Plane-parallel Stars

Stellar continua

Hydrogen and helium edges

$$\lambda_n \sim 91.16 \, \frac{n^2}{Z^2}$$

n		1	2	3	4	5	6	7	8	...
H I	n^2/Z^2	1	4	9	16	25	36	49	64	...
He II	n^2/Z^2	1/4	1	9/4	4	25/4	9	49/4	16	...

Balmer jump

F, G

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} = \frac{\sigma_\lambda(\mathrm{H}^-)\, N(\mathrm{H}^-)}{\sigma_\lambda(\mathrm{H}^-)\, N(\mathrm{H}^-) + \sigma_\lambda^{\mathrm{B}}\, N_{\mathrm{H}}(n=2)} < 1$$

F

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} \sim \frac{\sigma_\lambda(\mathrm{H}^-) N_{\mathrm{H}}(n=1)}{\sigma_\lambda^{\mathrm{B}} N_{\mathrm{H}}(n=2)}\, N_{\mathrm{e}}\, T_{\mathrm{e}}^{-3/2}\ \mathrm{e}^{h\nu/kT} \sim \frac{\sigma_\lambda(\mathrm{H}^-)}{\sigma_\lambda^{\mathrm{B}}}\, N_{\mathrm{e}}\, T_{\mathrm{e}}^{-3/2}\ \mathrm{e}^{2h\nu/kT}$$

O, B

$$\frac{\kappa(\lambda > 364.7)}{\kappa(\lambda < 364.7)} \sim \frac{\sigma_\lambda^{\mathrm{P}}\, N_{\mathrm{H}}(n=3)}{\sigma_\lambda^{\mathrm{B}}\, N_{\mathrm{H}}(n=2)} \sim \mathrm{e}^{-h\nu/kT_{\mathrm{e}}}$$

Lines from Plane-parallel Stars

Classical abundance determinations

Abundance

$$A_{\rm E} \equiv \frac{N_{\rm E}}{N_{\rm H}}$$

$$A_{12}({\rm E}) \equiv \log N_{\rm E} - \log N_{\rm H} + 12$$

$$[X] \equiv \log X_{\rm star} - \log X_{\rm Sun}$$

$$[{\rm Fe/H}] = \log(N_{\rm Fe}/N_{\rm H})_{\rm star} - \log(N_{\rm Fe}/N_{\rm H})_{\rm Sun}$$

$$n_l = b_l\, n_l^{\rm LTE} = b_l\, \frac{n_l^{\rm LTE}}{N_{\rm E}}\, N_{\rm H} A_{\rm E}$$

$$\alpha_\lambda^l = \frac{\sqrt{\pi} e^2}{m_{\rm e} c}\, \frac{\lambda^2}{c} b_l\, \frac{n_l^{\rm LTE}}{N_{\rm E}}\, N_{\rm H}\, A_{\rm E}\, f_{lu}\, \frac{H(a,v)}{\Delta\lambda_{\rm D}} \left[1 - \frac{b_u}{b_l}\, {\rm e}^{-hc/\lambda kT}\right]$$

Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Equivalent width

$$W_\lambda = \int_{\text{line}} \frac{I_c - I^l_\lambda}{I_c}\, \mathrm{d}\lambda$$

$$W_\lambda = \int_{\text{line}} \frac{\mathcal{F}_c - \mathcal{F}^l_\lambda}{\mathcal{F}_c}\, \mathrm{d}\lambda$$

Schuster-Schwarzschild atmosphere

$$I_\lambda = B_\lambda(T_\mathrm{R})\, \mathrm{e}^{-\tau_\lambda} + B_\lambda(T_\mathrm{L})(1 - \mathrm{e}^{-\tau_\lambda})$$

$$\tau_\lambda = \sigma_\lambda\, N_\mathrm{i} = \frac{\sqrt{\pi}\mathrm{e}^2}{m_\mathrm{e} c}\, \frac{\lambda_0^2}{c}\, \frac{f}{\Delta\lambda_\mathrm{D}}\, N_\mathrm{i}\, H(a,v) \approx \tau_{\lambda_0}\, H(a,v)$$

$$D_\lambda \equiv \frac{I_\mathrm{c} - I_\lambda}{I_\mathrm{c}} = \frac{B_\lambda(T_\mathrm{R}) - B_\lambda(T_\mathrm{L})}{B_\lambda(T_\mathrm{R})}(1 - \mathrm{e}^{-\tau_\lambda}) = D_\mathrm{max}(1 - \mathrm{e}^{-\tau_\lambda})$$

Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Schuster-Schwarzschild atmosphere

$$D_\lambda \equiv \frac{I_c - I_\lambda}{I_c} = \frac{B_\lambda(T_R) - B_\lambda(T_L)}{B_\lambda(T_R)} (1 - e^{-\tau_\lambda}) = D_{max}(1 - e^{-\tau_\lambda})$$

$$D_{max} \equiv \frac{B_\lambda(T_R) - B_\lambda(T_L)}{B_\lambda(T_R)}$$

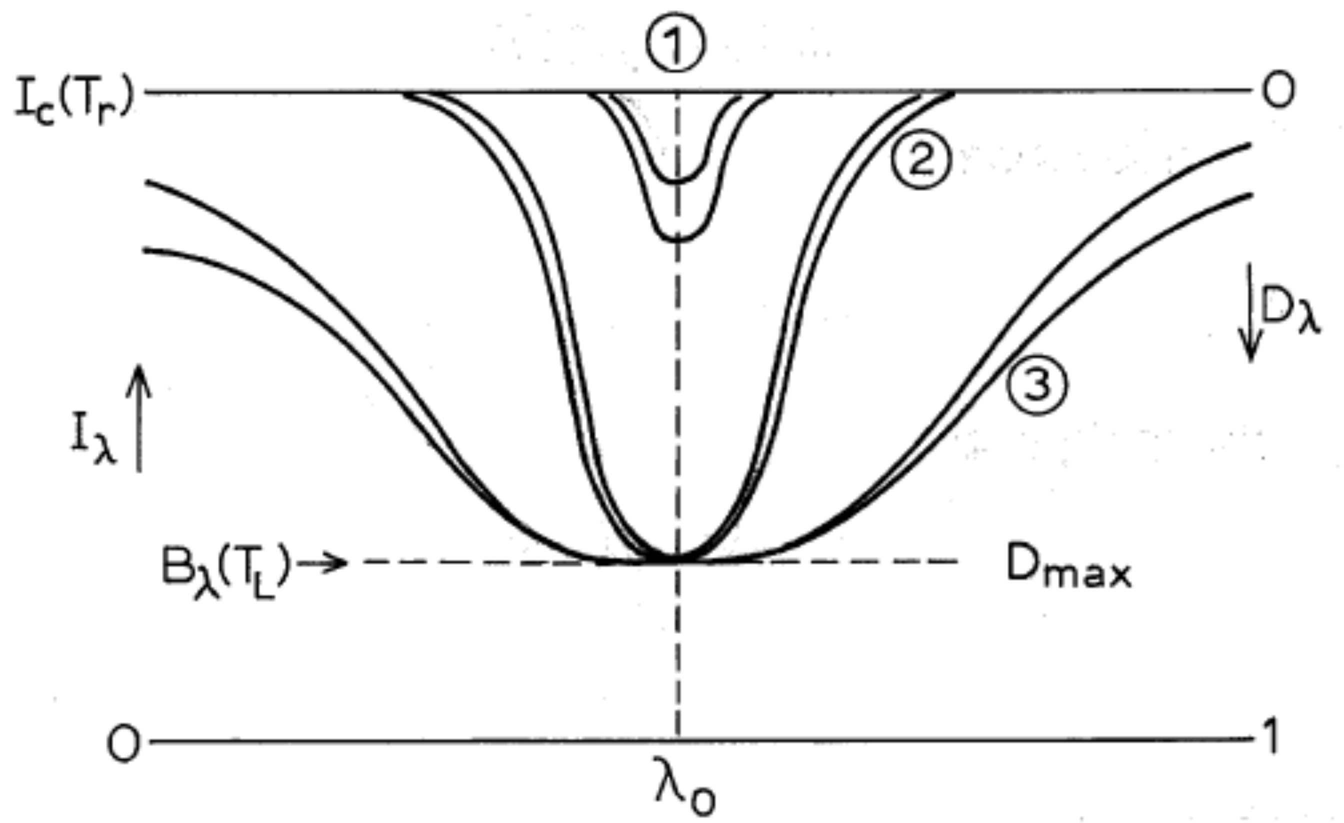
$$W_\lambda = D_{max} \int_{line} (1 - e^{-\tau_\lambda})\, d\lambda$$

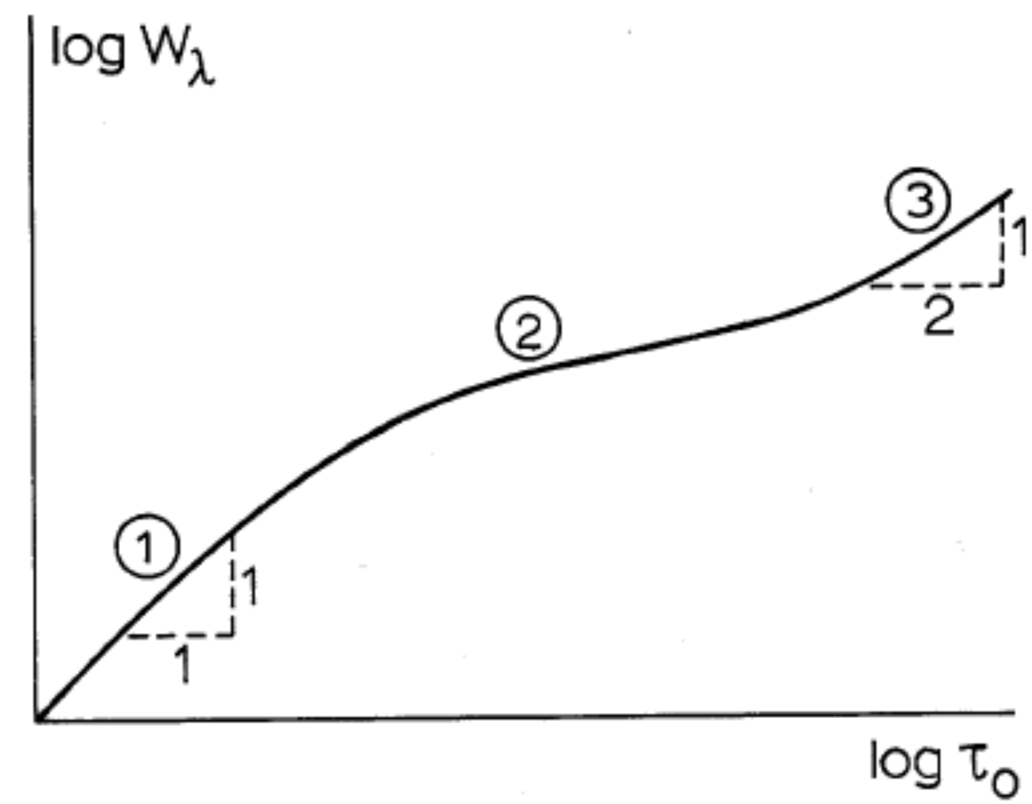
Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Schuster-Schwarzschild atmosphere





Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Weak lines

$$\tau_\lambda \ll 1$$

$$\exp(-\tau_\lambda) \approx 1 - \tau_\lambda$$

$$D_\lambda \approx D_{\rm max}\,\tau_\lambda$$

$$\begin{aligned} D_\lambda &\approx D_{\rm max}\,\tau_{\lambda_0}\ {\rm e}^{-(\Delta\lambda/\Delta\lambda_{\rm D})^2} \\ W_\lambda &\approx D_{\rm max}\tau_{\lambda_0}\sqrt{\pi}\,\Delta\lambda_{\rm D} = \frac{\pi {\rm e}^2}{m_e c}\frac{\lambda_0^2}{c}\,f\,D_{\rm max}\,N_{\rm i} \end{aligned}$$

Saturated lines

$$\tau_{\lambda_0} > 1$$

$$W_\lambda \approx Q\,D_{\rm max}\,\Delta\lambda_{\rm D}$$

$$Q = 2 - 4$$

Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Strong lines

$$\tau_{\lambda_0} \gg 1$$

$$H(a,v) \approx a/(\sqrt{\pi}\, v^2) = (a/\sqrt{\pi})\,(\Delta\lambda_{\rm D}/\Delta\lambda)^2 \sim 1/\Delta\lambda^2$$

$$\tau_\lambda = \tau_{\lambda_0} \frac{a}{\sqrt{\pi} v^2} = \tau_{\lambda_0} \frac{a}{\sqrt{\pi}} \frac{\Delta\lambda_{\rm D}^2}{\Delta\lambda^2}$$

$$u^2 = \Delta\lambda^2/(\tau_{\lambda_0}(a/\sqrt{\pi})\Delta\lambda_{\rm D}^2)$$

$$\begin{aligned} W_\lambda &= D_{\rm max} \int_{\rm line} (1 - {\rm e}^{-\tau_\lambda})\, {\rm d}\lambda \\ &= D_{\rm max} \Delta\lambda_{\rm D} \sqrt{\tau_{\lambda_0}(a/\sqrt{\pi})} \int_{\rm line} (1 - {\rm e}^{-1/u^2})\, {\rm d}u \\ &\sim D_{\rm max} \Delta\lambda_{\rm D} \sqrt{\tau_{\lambda_0}\, a} \end{aligned}$$

$$W_\lambda \sim \sqrt{\tau_{\lambda_0}\, a} \sim \sqrt{f\, N_i\, \gamma}$$

Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Milne-Eddington atmosphere

$$\eta_\lambda \equiv \kappa_\lambda^l / \kappa_\lambda^c$$

$$B_\lambda(\tau_c) = B_0 + b_c \tau_c$$

$$B_\lambda(\tau_\lambda) = B_0 + \frac{b_c}{1+\eta_\lambda}\tau_\lambda$$

$$F_\lambda(0) = B_0 + \frac{b_c}{1+\eta_\lambda}\frac{2}{3}$$

$$D_\lambda \equiv \frac{F_c(0) - F_\lambda(0)}{F_c(0)}$$

$$= \frac{(2/3)\, b_c\, \eta_\lambda/(1+\eta_\lambda)}{B_0 + (2/3) b_c}$$

$$= D_{\max} \frac{\eta_\lambda}{1+\eta_\lambda}$$

$$D_{\max} = (2/3) b_c / (B_0 + (2/3) b_c)$$

Lines from Plane-parallel Stars

Classical abundance determinations

Curve of growth methods

Milne-Eddington atmosphere

$$\eta_v = \eta_0 H(a, v)$$

$$\begin{aligned}
W_\lambda &= \int_{\rm line} D_\lambda \, {\rm d}\lambda = D_{\rm max} \, \Delta\lambda_{\rm D} \int_{\rm line} \frac{\eta_v}{1+\eta_v} \, {\rm d}v \\
\frac{W_\lambda}{D_{\rm max}\Delta\lambda_{\rm D}} &= \int_{\rm line} \frac{\eta_0 H(a,v)}{1+\eta_0 H(a,v)} \, {\rm d}v \\
\frac{W_\lambda}{D_{\rm max}\Delta\lambda_{\rm D}} &= \sqrt{\pi}\, \eta_0 \quad \text{for } \eta_0 \ll 1 \\
\frac{W_\lambda}{D_{\rm max}\Delta\lambda_{\rm D}} &= 2-4 \quad \text{for } \eta_0 > 1 \\
\frac{W_\lambda}{D_{\rm max}\Delta\lambda_{\rm D}} &= \sqrt{\pi^{3/2}\, a\, \eta_0} \quad \text{for } \eta_0 \gg 1.
\end{aligned}$$